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On Conditions for the Approximability of the Fundamental Groups of Graphs of Groups by Root Classes of Groups

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Abstract—Suppose that Γ is a non-empty connected graph, $\mathfrak{G}$ is the fundamental group of a graph of groups over Γ, and $\mathcal{C}$ is a root class of groups (the last means that $\mathcal{C}$ contains non-trivial groups and is closed under taking subgroups, extensions, and Cartesian powers of a certain type). It is known that $\mathfrak{G}$ is residually a $\mathcal{C}$-group if it has a homomorphism onto a group from $\mathcal{C}$ acting injectively on all vertex groups. We prove that, in this assertion, the words “vertex groups” can be replaced by “edge subgroups” provided all vertex groups are residually $\mathcal{C}$-groups. We also show that the converse doesn't need to hold if $\mathcal{C}$ consists of periodic groups and contains at least one infinite group.

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1. INTRODUCTION. STATEMENT OF RESULTS

This article continues the paper [1], where the relationship is considered between two properties of the fundamental group of a graph of groups. The first of these properties is the approximability by a root class $\mathcal{C}$, and the second is the existence of a homomorphism of the fundamental group that maps it onto a group from $\mathcal{C}$ and acts injectively on all edge subgroups.

Let us recall that a group X is said to be *approximable by a class of groups* $\mathcal{C}$ (this term was introduced by Malcev [2]) or *residually a $\mathcal{C}$-group* (this term belongs to Hall [3]) if, for each $x \in X \setminus \{1\}$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ (a *$\mathcal{C}$-group*) such that $x\sigma \neq 1$. The property of *residual finiteness* (i.e., the approximability by the class of all finite groups) is most famous because a finitely presented group with this property has a solvable word problem [4]. At the same time, the approximability by other classes of groups is also considered in the literature, and many of these classes are root classes of groups.

In accordance with one of the equivalent definitions (see Proposition 3.2 below), a class of groups $\mathcal{C}$ is called a *root class* if it contains non-trivial groups and is closed under taking subgroups, extensions, and Cartesian products of the form $\prod_{y\in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$. The concept of a root class was introduced by K. Gruenberg [3] and turned out to be very useful in studying the approximability of the fundamental groups of various graphs of groups [5–14]. Thanks to its use, it became possible, in particular, to make significant progress in the study of the residual p-finiteness (where p is a prime number) and the residual solvability of such groups.

Everywhere below, it is assumed that $\Gamma = (\mathcal{V}, \mathcal{E})$ is a non-empty connected undirected graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ (loops and multiple edges are allowed). We define a directed *graph of groups*

$$\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in \mathcal{V}),\ H_e,\ \varphi_{\pm e}\ (e \in \mathcal{E})\big)$$

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over Γ by assigning to each vertex $v \in \mathcal{V}$ some group G_v, and to each edge $e \in \mathcal{E}$ a direction, a group H_e, and injective homomorphisms $\varphi_{+e}\colon H_e \to G_{e(1)}$, $\varphi_{-e}\colon H_e \to G_{e(-1)}$ (where $e(1)$ and $e(-1)$ denote the vertices that are the ends of e). Let us refer to the groups G_v and the subgroups $H_{+e} = H_e\varphi_{+e}$, $H_{-e} = H_e\varphi_{-e}$ as *vertex groups* and *edge subgroups*, respectively.

For each maximal tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}_{\mathcal{T}})$ in Γ, we can consider a group representation whose generators are the generators of the groups G_v $(v \in \mathcal{V})$ and symbols t_e $(e \in \mathcal{E} \setminus \mathcal{E}_{\mathcal{T}})$, and whose defining relations are the relations of G_v $(v \in \mathcal{V})$ and all possible relations of the form

$$h\varphi_{+e} = h\varphi_{-e}\ (e \in \mathcal{E}_{\mathcal{T}},\ h \in H_e), \quad t_e^{-1}(h\varphi_{+e})t_e = h\varphi_{-e}\ (e \in \mathcal{E} \setminus \mathcal{E}_{\mathcal{T}},\ h \in H_e),$$

where $h\varphi_{\varepsilon e}$ $(\varepsilon = \pm 1)$ is a word in the generators of $G_{e(\varepsilon)}$ that represents the image of h under $\varphi_{\varepsilon e}$. It is known ([15], Sect. 5.1) that, for all maximal trees in Γ, the corresponding representations of the described form determine, up to isomorphism, the same group. This group is called the *fundamental group* of the graph of groups $\mathcal{G}(\Gamma)$ and is usually denoted by $\pi_1(\mathcal{G}(\Gamma))$. It is also known ([15], Sect. 5.2) that the identity mappings of generators determine embeddings of the vertex groups G_v $(v \in \mathcal{V})$ into $\pi_1(\mathcal{G}(\Gamma))$, and therefore these groups can be considered subgroups of $\pi_1(\mathcal{G}(\Gamma))$. Let us write $\pi_1(\mathcal{G}(\Gamma), \mathcal{T})$ to specify the maximal tree $\mathcal{T}$ used to construct a representation of $\pi_1(\mathcal{G}(\Gamma))$.

An important role in the study of the approximability of $\pi_1(\mathcal{G}(\Gamma))$ by a root class $\mathcal{C}$ belongs to homomorphisms that map this group onto $\mathcal{C}$-groups and act injectively on all its vertex groups (for brevity, we call them below *homomorphisms of type* (i)). If such a homomorphism exists, then $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group ([10], Proposition 7). But at the same time, all vertex groups turn out to be $\mathcal{C}$-groups, and thus we are dealing with a very special case. In the general case, when the vertex groups need not belong to $\mathcal{C}$, the so-called "filtration approach" is most often used to prove that $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group. This method was proposed by G. Baumslag [16] and consists in the approximability of $\pi_1(\mathcal{G}(\Gamma))$ by the fundamental groups of graphs of groups having homomorphisms of type (i). For all its productivity, this approach also has certain limitations, a detailed discussion of which is given in [10]. Therefore, the problem arises of finding conditions of a general form which make it possible to establish the approximability of $\pi_1(\mathcal{G}(\Gamma))$ without using the filtration method. In the present article, the following theorem is proved in this direction.

Theorem 1. *Suppose that $\mathcal{C}$ is a root class of groups and all vertex groups G_v $(v \in \mathcal{V})$ are residually $\mathcal{C}$-groups. If there exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a group from $\mathcal{C}$ acting injectively on all edge subgroups $H_{\varepsilon e}$ $(e \in \mathcal{E}, \varepsilon = \pm 1)$, then $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group.*

The above theorem generalizes Theorem 1 from [1], which says that the same assertion holds if Γ is a finite graph. It becomes possible to discard this finiteness condition due to a completely different proof. The latter is based on two facts on the structure of subgroups of $\pi_1(\mathcal{G}(\Gamma))$, which are of independent interest (see Propositions 2.4 and 3.4 below).

The map σ appearing in Theorem 1 is referred to below as a *homomorphism of type* (ii). Let us note that, if a more complex free construction is composed of several simpler ones and we want to prove the existence of a homomorphism of type (i) of the complex construction, then it may not be enough for the simple constructions to have homomorphisms of the same type. However, if we replace (i) with (ii), the situation improves: this can be seen, for example, from the proof of Theorem 2 in [17]. Thus, the search for conditions for the existence of homomorphisms of type (ii) is an independent problem, the results of which can be used both for direct proof of approximability and for the subsequent application of the filtration approach.

Now let us turn to the question of whether the converse of Theorem 1 holds. The following assertion is proved in [1].

Theorem 2. *If $\mathcal{C}$ is a root class of groups containing at least one infinite group and not containing some (absolutely) free group of finite or countable rank, then for any graph $\Gamma = (\mathcal{V}, \mathcal{E})$, there exists a graph of groups $\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in \mathcal{V}),\ H_e,\ \varphi_{\pm e}\ (e \in \mathcal{E})\big)$ such that:*

1) all groups G_v $(v \in \mathcal{V})$ are residually $\mathcal{C}$-groups;

2) all subgroups $H_{\varepsilon e}$ $(e \in \mathcal{E}, \varepsilon = \pm 1)$ belong to $\mathcal{C}$ and are non-trivial;

3) $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group;

4) if $\mathcal{T}$ is a maximal tree in Γ and σ is a homomorphism of $\pi_1(\mathcal{G}(\Gamma), \mathcal{T})$ onto a group from $\mathcal{C}$, then for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, the relation $\ker\sigma \cap H_{\varepsilon e} \neq 1$ holds.

The above theorem says that, in very many cases, the approximability of $\pi_1(\mathcal{G}(\Gamma))$ by a root class $\mathcal{C}$ does not imply that this group has a homomorphism of type (ii). However, its proof given in [1] makes essential use of the fact that the vertex groups of the constructed graph of groups do not have homomorphisms onto $\mathcal{C}$-groups that are injective on their edge subgroups. Thus, the absence of a homomorphism of type (ii) is explained not so much by the structure of the group $\pi_1(\mathcal{G}(\Gamma))$ as a whole, but by the properties of its vertex groups. In the present article, we consider the case when all vertex groups belong to $\mathcal{C}$ and, therefore, cannot interfere with the existence of a homomorphism of the indicated type. The following assertion is proved.

Theorem 3. *If $\mathcal{C}$ is a root class of groups consisting of periodic groups and containing at least one infinite group, then for any graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with a non-empty set of edges, there exists a graph of groups*

$$\mathcal{G}(\Gamma) = \big(\Gamma,\ G_v\ (v \in \mathcal{V}),\ H_e,\ \varphi_{\pm e}\ (e \in \mathcal{E})\big)$$

such that:

1) all groups G_v $(v \in \mathcal{V})$ belong to the class $\mathcal{C}$;

2) all subgroups $H_{\varepsilon e}$ $(e \in \mathcal{E}, \varepsilon = \pm 1)$ are non-trivial;

3) $\pi_1(\mathcal{G}(\Gamma))$ is residually a $\mathcal{C}$-group;

4) if $\mathcal{T}$ is a maximal tree in Γ and σ is a homomorphism of $\pi_1(\mathcal{G}(\Gamma), \mathcal{T})$ onto a group from $\mathcal{C}$, then for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, the relation $\ker\sigma \cap H_{\varepsilon e} \neq 1$ holds.

It remains an open question: whether an analog of Theorem 3 holds if $\mathcal{C}$ consists of finite groups or contains non-periodic groups.

2. ON SOME FREE CONSTRUCTIONS OF GROUPS AND SUBGROUPS OF THESE CONSTRUCTIONS

Let us recall that if $\mathcal{E} = \{e\}$ and $e(1) \neq e(-1)$, then the fundamental group $\pi_1(\mathcal{G}(\Gamma))$ is said to be the *generalized free product* of the groups $G_{e(1)}$ and $G_{e(-1)}$ with the *amalgamated subgroup* $H_{+e} = H_{-e}$. If $\mathcal{V} = \{v\}$ and $\mathcal{E} \neq \varnothing$, then $\pi_1(\mathcal{G}(\Gamma))$ is the *HNN-extension* of the group G_v with the family of *stable letters* $\{t_e | e \in \mathcal{E}\}$ and G_v is the *base group* of this HNN-extension. Finally, if Γ is a tree, then $\pi_1(\mathcal{G}(\Gamma))$ is called the *tree product* of the groups G_v $(v \in \mathcal{V})$. Below, we also use the construction of the (ordinary) *free product* of a family of groups. Its definition and properties can be found, for example, in [18, § 6.2].

The next proposition follows from the results of [19].

Proposition 2.1. *If $\mathcal{V} = \{v\}$, $\mathcal{E} \neq \varnothing$, and N is a normal subgroup of $\pi_1(\mathcal{G}(\Gamma))$ that trivially intersects all edge subgroups $H_{\varepsilon e}$ $(e \in \mathcal{E}, \varepsilon = \pm 1)$, then N splits as the (ordinary) free product of a free group and groups, each of which is isomorphic to the subgroup $N \cap G_v$.*

Proposition 2.2 [20]. *If $\mathbb{P}$ is a free product of groups A_λ $(\lambda \in \Lambda)$ and N is a subgroup of $\mathbb{P}$, then N splits as the free product of a free group and groups, each of which is isomorphic to the subgroup $x^{-1}Nx \cap A_\lambda$ for some $x \in \mathbb{P}$, $\lambda \in \Lambda$.*

Proposition 2.3 ([10], Proposition 1). *Suppose that Δ is a non-empty connected subgraph of Γ and $\mathcal{G}(\Delta)$ is the graph of groups whose vertices and edges are associated with the same groups, directions, and homomorphisms as in $\mathcal{G}(\Gamma)$. If $\mathcal{T}$ is a maximal tree in Γ such that $\Delta \cap \mathcal{T}$ is a maximal tree in Δ, then the identity mapping of the generators of $\pi_1(\mathcal{G}(\Delta), \Delta \cap \mathcal{T})$ into $\pi_1(\mathcal{G}(\Gamma), \mathcal{T})$ defines an injective homomorphism.*

Proposition 2.4. *If Γ is a tree, then $\pi_1(\mathcal{G}(\Gamma))$ can be embedded in the HNN-extension*

$$\mathbb{E} = \big\langle G_v\ (v \in \mathcal{V}),\ t_e\ (e \in \mathcal{E});\ t_e^{-1} h\varphi_{+e} t_e = h\varphi_{-e}\ (e \in \mathcal{E},\ h \in H_e)\big\rangle.$$

Proof. It is well known that the normal closure of the base group of an HNN-extension is a tree product of isomorphic copies of this group. Since the base group $\mathbb{P}$ of the HNN-extension $\mathbb{E}$ is the free product of the groups G_v $(v \in \mathcal{V})$, its normal closure turns out to be a “forest” product (which corresponds in the general case to a forest, not a tree) of isomorphic copies of the groups G_v $(v \in \mathcal{V})$. We describe in more detail how this product is structured and then indicate a subtree whose fundamental group is isomorphic to $\pi_1(\mathcal{G}(\Gamma))$.

Let T be a free group with basis $\{t_e | e \in \mathcal{E}\}$. Consider the graph Γ' with the set of vertices $\mathcal{V}' = \mathcal{V} \times T$ and the set of edges $\mathcal{E}'$ indexed by the set $\mathcal{E} \times T$ and defined as follows: for any $e \in \mathcal{E}$, $t \in T$, the edge e' with index (e, t) connects the vertices $e'(1) = \big(e(1),\, t_e t\big)$ and $e'(-1) = \big(e(-1),\, t\big)$. It is easy to see that Γ' has no multiple edges or loops. To prove the acyclicity of Γ', we show that if L is a simple chain in this graph which has a non-zero length and joins a vertex $u' = (u, r)$ to a vertex $w' = (w, s)$, then there exists an element $\tau \in T \setminus \{1\}$ such that $\tau r = s$ and, therefore, $u' \neq w'$.

Indeed, let e' be the edge of L connecting the vertices $\big(e(1),\, t_e t\big)$ and $\big(e(-1),\, t\big)$ for some $e \in \mathcal{E}$, $t \in T$. If the movement along the chain (from u' to w') includes a transition from $\big(e(1),\, t_e t\big)$ to $\big(e(-1),\, t\big)$, then we associate e' with the element $t_{e'} = t_e^{-1}$, otherwise we put $t_{e'} = t_e$. Let us denote by τ the product of all elements corresponding to the edges of L and taken in the opposite order to the movement indicated above. Since L is a simple chain, its adjacent edges are associated with elements that are not mutually inverse. Hence, τ (as an element of T) has a reduced form of non-zero length and, therefore, is non-trivial ([18], § 2.1). The equality $\tau r = s$ follows from the definition of the elements $t_{e'}$.

For any $v \in \mathcal{V}$, $t \in T$, we denote by $G_{v,t}$ an isomorphic copy of G_v and by $\iota_{v,t}$ the isomorphism $G_v \to G_{v,t}$. Let $\mathcal{G}'(\Gamma')$ be the graph of groups such that the group $G_{v,t}$ is assigned to the vertex $(v, t) \in \mathcal{V}'$ ($v \in \mathcal{V}$, $t \in T$), while the group H_e and the homomorphisms

$$\varphi_{+e'} = \varphi_{+e}\iota_{e(1),\, t_e t}, \quad \varphi_{-e'} = \varphi_{-e}\iota_{e(-1),\, t}$$

are assigned to the edge $e' \in \mathcal{E}'$ with index (e, t) ($e \in \mathcal{E}$, $t \in T$). For any element $\tau \in T$, we consider the mapping of the vertices of Γ' defined by the rule $(v, t) \mapsto (v, t\tau)$ and the corresponding isomorphisms of the vertex groups $\iota_{v,t}^{-1}\iota_{v,t\tau}$. It is easy to see that these mappings induce automorphisms of Γ' and $\mathcal{G}'(\Gamma')$, and, hence, an automorphism α_τ of $\pi_1(\mathcal{G}'(\Gamma'))$. It is also obvious that $\alpha_{\tau_1\tau_2} = \alpha_{\tau_1}\alpha_{\tau_2}$ for any $\tau_1, \tau_2 \in T$. Therefore, we can consider the split extension S of $\pi_1(\mathcal{G}'(\Gamma'))$ by T such that the conjugation by $\tau \in T$ acts on $\pi_1(\mathcal{G}'(\Gamma'))$ as α_τ.

The group S has the representation

$$\left\langle \begin{matrix} G_{v,t}\ (v \in \mathcal{V},\ t \in T), \\ t_e\ (e \in \mathcal{E}) \end{matrix} \;\middle|\; \begin{matrix} \tau^{-1} g \tau = g\alpha_\tau\ (g \in G_{v,t},\ v \in \mathcal{V},\ t, \tau \in T), \\ h\varphi_{+e}\iota_{e(1),\, t_e t} = h\varphi_{-e}\iota_{e(-1),\, t}\ (e \in \mathcal{E},\ h \in H_e,\ t \in T) \end{matrix} \right\rangle.$$

Since, for any $v \in \mathcal{V}$, $g \in G_v$, $\tau \in T$, the equalities $\tau^{-1} g\iota_{v,1} \tau = g\iota_{v,1}\alpha_\tau = g\iota_{v,1}\iota_{v,1}^{-1}\iota_{v,\tau} = g\iota_{v,\tau}$ hold in S, the generators of $G_{v,t}$ ($v \in \mathcal{V}$, $t \in T \setminus \{1\}$) can be excluded from this representation together with the relations

$$\tau^{-1} g \tau = g\alpha_\tau \quad (g \in G_{v,1},\ v \in \mathcal{V},\ \tau \in T).$$

As a result, the relations

$$\tau^{-1} g \tau = g\alpha_\tau \quad (g \in G_{v,t},\ v \in \mathcal{V},\ t \in T \setminus \{1\},\ \tau \in T)$$

turn into identities, the relations

$$h\varphi_{+e}\iota_{e(1),\, t_e t} = h\varphi_{-e}\iota_{e(-1),\, t} \quad (e \in \mathcal{E},\ h \in H_e,\ t \in T)$$

take the form

$$t_e^{-1}(h\varphi_{+e}\iota_{e(1),1})t_e = h\varphi_{-e}\iota_{e(-1),1} \quad (e \in \mathcal{E},\ h \in H_e),$$

and the representation of S can be turned into the representation of $\mathbb{E}$ by identifying the elements of G_v ($v \in \mathcal{V}$) and their images under $\iota_{v,1}$. Therefore, $\mathbb{E} \cong S$.

Now let us build an embedding of Γ into Γ'. To do this, we fix some vertex $u \in \mathcal{V}$ and argue by induction on the length of a (unique) path in the tree Γ joining an arbitrarily chosen vertex $v \in \mathcal{V}$ to u.

Each vertex $v \in \mathcal{V}$ will be mapped to a vertex of the form (v, t) for some $t \in T$. Let us associate u with $(u, 1)$. If $v \in \mathcal{V}$ is a vertex other than u, e is the last edge of the path joining u to v, $\varepsilon = \pm 1$ is the number satisfying the equality $v = e(\varepsilon)$, and the vertex $e(-\varepsilon)$ corresponds to the vertex $\big(e(-\varepsilon),\, t\big)$ for some $t \in T$, then we associate the vertex $v = e(\varepsilon)$ with $\big(e(\varepsilon),\, t_e^\varepsilon t\big)$. Let e' be the edge of Γ' with index (e, t) if $\varepsilon = 1$, or with index $\big(e,\, t_e^{-1} t\big)$ if $\varepsilon = -1$. Then, the equalities $e'(\varepsilon) = \big(e(\varepsilon),\, t_e^\varepsilon t\big)$, $e'(-\varepsilon) = \big(e(-\varepsilon),\, t\big)$

hold by the definition of Γ'. Therefore, the constructed mapping of vertices defines the desired embedding of Γ into Γ'. It is also easy to see that, in combination with the isomorphisms $\iota_{v,t}$ ($v \in \mathcal{V}$, $t \in T$), it determines an embedding of $\mathcal{G}(\Gamma)$ into $\mathcal{G}'(\Gamma')$. Thus, by Proposition 2.3, the group $\pi_1(\mathcal{G}(\Gamma))$ can be embedded into the group $\pi_1(\mathcal{G}'(\Gamma'))$, as required. □

3. PROOF OF THEOREM 1

Given a class of groups $\mathcal{C}$ and a group X, we denote by $\mathcal{C}^*(X)$ the family of normal subgroups of X such that $Y \in \mathcal{C}^*(X)$ if and only if $X/Y \in \mathcal{C}$.

Proposition 3.1. *If $\mathcal{C}$ is a class of groups closed under taking subgroups, X is a group, Y and Z are its subgroups, and $Y \in \mathcal{C}^*(X)$, then $Y \cap Z \in \mathcal{C}^*(Z)$.*

Proof. Indeed, $Z/Y \cap Z \cong ZY/Y \leqslant X/Y \in \mathcal{C}$ and $Z/Y \cap Z \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking subgroups. □

Proposition 3.2 [21, Theorem 1]**.** *If $\mathcal{C}$ is a class of groups closed under taking subgroups, then the following statements are equivalent.*

1. The class $\mathcal{C}$ satisfies the Gruenberg condition*: for any group X and for any subnormal series $1 \leqslant Z \leqslant Y \leqslant X$ whose factors X/Y and Y/Z belong to $\mathcal{C}$, there exists a subgroup $T \in \mathcal{C}^*(X)$ such that $T \leqslant Z$.*

2. The class $\mathcal{C}$ is closed under taking Cartesian wreath products.

3. The class $\mathcal{C}$ is closed under taking extensions and, for any two groups $X, Y \in \mathcal{C}$, contains the Cartesian product $\prod_{y \in Y} X_y$, where X_y is an isomorphic copy of X for each $y \in Y$.

Proposition 3.3. *If $\mathcal{C}$ is a root class of groups, then the following statements hold.*

1. Every free group is residually a $\mathcal{C}$-group ([5], Theorem 1).

2. The free product of any number of residually $\mathcal{C}$-groups is residually a $\mathcal{C}$-group ([3], Theorem 4.1; [5], Theorem 2).

3. Any extension of a residually $\mathcal{C}$-group by a $\mathcal{C}$-group is again residually a $\mathcal{C}$-group ([3], Lemma 1.5).

For any family of groups Ω, we denote by $\mathcal{P}(\Omega)$ the class of groups consisting of (ordinary) free products, each factor of which is a free group or can be embedded in a group from Ω. It follows from Proposition 2.2 that this class is closed under taking subgroups. Therefore, if Θ is a family of $\mathcal{P}(\Omega)$-groups, then $\mathcal{P}(\Theta) \subseteq \mathcal{P}(\Omega)$.

Proposition 3.4. *Suppose that $\mathcal{C}$ is a root class of groups and there exists a homomorphism σ of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group that acts injectively on all edge subgroups $H_{\varepsilon e}$ ($e \in \mathcal{E}$, $\varepsilon = \pm 1$). If $N = ker\sigma$ and $\Omega = \{N \cap G_v | v \in \mathcal{V}\}$, then $\pi_1(\mathcal{G}(\Gamma))$ is an extension of a $\mathcal{P}(\Omega)$-group by a $\mathcal{C}$-group.*

Proof. Assume first that Γ is a tree, and consider the HNN-extension $\mathbb{E}$ from Proposition 2.4. Obviously, $\mathbb{T} = \pi_1(\mathcal{G}(\Gamma))$ is the quotient group of $\mathbb{E}$ by the normal closure of the set $\{t_e | e \in \mathcal{E}\}$. Let us denote by K the preimage of N under the natural homomorphism $\varepsilon \colon \mathbb{E} \to \mathbb{T}$. Then $K \in \mathcal{C}^*(\mathbb{E})$ and, because ε acts on every vertex group of $\mathbb{E}$ as the identity mapping, the equality $K \cap G_v = N \cap G_v$ holds for any $v \in \mathcal{V}$. This implies, in particular, that $K \cap H_{\varepsilon e} = 1$ for all $e \in \mathcal{E}$, $\varepsilon = \pm 1$. Hence, by Proposition 2.1, $K \in \mathcal{P}\big(\{K \cap \mathbb{P}\}\big)$, where $\mathbb{P}$ is the base group of $\mathbb{E}$, i.e., the free product of groups G_v ($v \in \mathcal{V}$). According to Proposition 2.2, $K \cap \mathbb{P} \in \mathcal{P}(\Theta)$, where $\Theta = \big\{(K \cap \mathbb{P}) \cap G_v | v \in \mathcal{V}\big\}$. Since $\Theta = \big\{K \cap G_v | v \in \mathcal{V}\big\} = \Omega$, we have $K \cap \mathbb{P} \in \mathcal{P}(\Omega)$ and $K \in \mathcal{P}(\Omega)$. By Proposition 2.4, $\mathbb{T}$ can be considered a subgroup of $\mathbb{E}$ and, therefore, turns out to be an extension of $\mathbb{T} \cap K$ by $\mathbb{T}/\mathbb{T} \cap K$. It remains to note that $\mathbb{T} \cap K \in \mathcal{P}(\Omega)$ and $\mathbb{T}/\mathbb{T} \cap K \in \mathcal{C}$: this follows from Proposition 3.1 and the fact that the classes $\mathcal{P}(\Omega)$ and $\mathcal{C}$ are closed under taking subgroups.

So, if Γ is a tree, the proposition is proved. Assume now that Γ is an arbitrary connected graph and $\mathcal{T}$ is a maximal tree in Γ used to construct a representation of $\pi_1(\mathcal{G}(\Gamma))$. Since the restriction $\sigma_{\mathcal{T}}$ of σ to the tree product $\pi_1(\mathcal{G}(\mathcal{T}))$ acts injectively on all edge subgroups of this product and

$$\big\{\ker\sigma_{\mathcal{T}} \cap G_v | v \in \mathcal{V}\big\} = \Omega,$$

it follows from the above that $\pi_1(\mathcal{G}(\mathcal{T}))$ is an extension of some $\mathcal{P}(\Omega)$-group P by a group from $\mathcal{C}$. Let us put $U = N \cap \pi_1(\mathcal{G}(\mathcal{T}))$ and $V = U \cap P$. Then, $V \in \mathcal{C}^*(U) \cap \mathcal{P}(\Omega)$ because the classes $\mathcal{P}(\Omega)$ and $\mathcal{C}$ are closed under taking subgroups and Proposition 3.1 can be applied to the subgroups $P \in \mathcal{C}^*(\pi_1(\mathcal{G}(\mathcal{T})))$ and U.

Since $\pi_1(\mathcal{G}(\Gamma))$ is an HNN-extension of $\pi_1(\mathcal{G}(\mathcal{T}))$, the subgroup N splits, by Proposition 2.1, as the free product of a free group F and groups X_i ($i \in \mathcal{I}$) isomorphic to U. Let $\theta\colon N \to U$ be the surjective homomorphism extending the isomorphisms $X_i \to U$ ($i \in \mathcal{I}$) and taking F to 1. Let also $M = \ker\theta\delta$, where $\delta\colon U \to U/V$ is a natural homomorphism. Then, $M \in \mathcal{C}^*(N)$ and $M \cap X_i \cong V \in \mathcal{P}(\Omega)$ for all $i \in \mathcal{I}$. By applying Proposition 2.2 to the free product N, we get

$$M \in \mathcal{P}\big(\{M \cap X_i | i \in \mathcal{I}\}\big).$$

Hence, $M \in \mathcal{P}(\Omega)$.

Since $M \leqslant N \leqslant \pi_1(\mathcal{G}(\Gamma))$ is a subnormal sequence whose factors belong to $\mathcal{C}$, it follows from Proposition 3.2 that M contains a subgroup $L \in \mathcal{C}^*(\pi_1(\mathcal{G}(\Gamma)))$. Because the class $\mathcal{P}(\Omega)$ is closed under taking subgroups, we have $L \in \mathcal{P}(\Omega)$. Therefore, $\pi_1(\mathcal{G}(\Gamma))$ is an extension of the $\mathcal{P}(\Omega)$-group L by the $\mathcal{C}$-group $\pi_1(\mathcal{G}(\Gamma))/L$. □

Proof of Theorem 1. Proposition 3.3 implies that if $\mathcal{C}$ is a root class of groups and a family Ω consists of residually $\mathcal{C}$-groups, then any extension of a $\mathcal{P}(\Omega)$-group by a $\mathcal{C}$-group is residually a $\mathcal{C}$-group. Therefore, Theorem 1 immediately follows from Proposition 3.4. □

4. PROOF OF THEOREM 3

Proposition 4.1. *Suppose that p is a prime number and $n = p^l$ for some $l \geqslant 1$. Suppose also that λ and μ are the bijections of the set $M = \{0, 1, \ldots, n-1\}$ defined as follows*

$$\lambda(i) = (i+1) \bmod n, \quad \mu(i) = \begin{cases} i+1, & i \not\equiv p-1 \pmod p, \\ i-(p-1), & i \equiv p-1 \pmod p. \end{cases}$$

If X denotes the subgroup of the group of bijective mappings of M generated by the elements λ and μ, then $X^n = 1$ and, therefore, X is a finite p-group.

Proof. Let x be an element of X written as a product of the generators λ and μ. Let also $\sigma_\lambda(x)$ and $\sigma_\mu(x)$ denote the sums of exponents of λ and μ in this product. It is easy to see that, for any $i \in M$,

$$\lambda(i) = \begin{cases} \mu(i), & i \not\equiv p-1 \pmod p, \\ (\mu\lambda^p)(i), & i \equiv p-1 \pmod p, \end{cases} \qquad \lambda^{-1}(i) = \begin{cases} \mu^{-1}(i), & i \not\equiv 0 \pmod p, \\ (\lambda^{-p}\mu^{-1})(i), & i \equiv 0 \pmod p. \end{cases}$$

It follows that, for each $i \in M$, there exist an element $y_i \in \operatorname{sgp}\{\lambda^p, \mu\}$ and a number $k_i \in \mathbb{Z}$ such that

$$(x^p)(i) = y_i(i), \quad \sigma_\mu(y_i) = p\big(\sigma_\lambda(x) + \sigma_\mu(x)\big), \quad \sigma_\lambda(y_i) = pk_i$$

(here, as above, $\sigma_\lambda(y_i)$ and $\sigma_\mu(y_i)$ denote the sums of exponents of λ and μ in the fixed representation of y_i as a product of generators). Since $\mu^p = [\lambda^p, \mu] = 1$, the equality $y_i = \lambda^{pk_i}$ holds. Therefore,

$$(x^p)(i) = (\lambda^{pk_i})(i) = (i + pk_i) \bmod n.$$

If $j = (i + pr) \bmod n$ for some $r \in \mathbb{Z}$, then

$$(x^p)(j) = (\lambda^{pr}x^p)(i) = (x^p\lambda^{pr})(i) = \big(i + p(k_i + r)\big) \bmod n.$$

Using these relations and obvious induction, we get that, for any $s \geqslant 1$, the equality

$$(x^p)^s(i) = (i + psk_i) \bmod n$$

holds. Therefore, $(x^p)^{n/p}(i) = i$, and since x and i are chosen arbitrarily, $X^n = 1$. □

Proposition 4.2. *Suppose that p, n, M, λ, and μ are defined in the same way as in Proposition 4.1. Suppose also that $C_0, \ldots, C_{n-1}$ are cyclic groups of order p with generators $c_0, \ldots, c_{n-1}$ respectively; H_n is the direct product of the groups C_i ($i \in M$); α_n and β_n are the automorphisms of H_n acting according to the rule*

$$c_i\alpha_n = c_{\mu(i)}, \quad c_i\beta_n = c_{(\lambda^{-1}\mu\lambda)(i)} \quad (i \in M);$$

A_n and B_n are the split extensions of H_n by the cyclic groups $\langle\alpha_n\rangle$ and $\langle\beta_n\rangle$, respectively. Then, the generalized free product P_n of the groups A_n and B_n with the amalgamated subgroup H_n is residually p-finite.

Proof. Since H_n is normal in A_n and B_n, we can consider the group $\mathrm{Aut}_{P_n}(H_n)$ consisting of the restrictions on H_n of all inner automorphisms of P_n. Obviously, $\mathrm{Aut}_{P_n}(H_n)$ is generated by α_n and β_n and is isomorphic to the subgroup of the group X from Proposition 4.1 generated by the bijections μ and $\lambda^{-1}\mu\lambda$. It follows that $\mathrm{Aut}_{P_n}(H_n)$ is a finite p-group. Since A_n and B_n are also finite p-groups, P_n is residually p-finite by ([22], Corollary 2). □

Proposition 4.3. *Suppose that p is a prime number, λ_∞ and μ_∞ are the bijections of $\mathbb{Z}$ defined as follows*

$$\lambda_\infty(i) = i+1, \quad \mu_\infty(i) = \begin{cases} i+1, & i \not\equiv p-1 \pmod p, \\ i-(p-1), & i \equiv p-1 \pmod p. \end{cases}$$

Suppose also that C_i is a cyclic group of order p with a generator c_i for any $i \in \mathbb{Z}$; H_∞ is the direct product of the groups C_i $(i \in \mathbb{Z})$; α_∞ and β_∞ are the automorphisms of H_∞ acting according to the rule

$$c_i\alpha_\infty = c_{\mu_\infty(i)}, \quad c_i\beta_\infty = c_{(\lambda_\infty^{-1}\mu_\infty\lambda_\infty)(i)} \quad (i \in \mathbb{Z});$$

A_∞ and B_∞ are the split extensions of H_∞ by the cyclic groups $\langle\alpha_\infty\rangle$ and $\langle\beta_\infty\rangle$, respectively. Then the following statements hold.

1. The generalized free product P_∞ of the groups A_∞ and B_∞ with the amalgamated subgroup H_∞ is residually p-finite.

2. The automorphism $\alpha_\infty^{-1}\beta_\infty$ has an infinite order.

Proof. 1. Let us take an element $x \in P_\infty \setminus \{1\}$ and find a homomorphism of P_∞ onto a finite p-group that maps x to a non-trivial element.

The subgroup H_∞ is normal in P_∞, and the quotient group P_∞/H_∞ splits as the (ordinary) free product of two finite p-groups $\langle\alpha_\infty\rangle$ and $\langle\beta_\infty\rangle$. Therefore, if $x \notin H_\infty$, then the natural homomorphism $P_\infty \to P_\infty/H_\infty$ can be extended to the desired one by Proposition 3.3.

If $x \in H_\infty$, then there exists a p-number n such that

$$x \in \prod_{-n/2\leqslant i<n/2} C_i.$$

Let us consider the mapping of the generators of P_∞ into the group P_n from Proposition 4.2 acting according to the rule

$$\alpha_\infty \mapsto \alpha_n, \quad \beta_\infty \mapsto \beta_n, \quad c_i \mapsto c_{i \bmod n}.$$

Since $p|n$, this mapping defines a homomorphism, which we denote by σ. It is clear that $x \notin \ker\sigma$ due to the choice of n. Therefore, σ can be extended to the desired mapping.

2. It can be directly verified that if $i \in \mathbb{Z}$ and $i \equiv 0 \pmod p$, then $[\mu_\infty, \lambda_\infty](i) = i+p$. Hence, $c_i(\alpha_\infty^{-1}\beta_\infty) = c_{i+p}$ and, therefore, the order of $\alpha_\infty^{-1}\beta_\infty$ is infinite. □

Proof of Theorem 3. Since $\mathcal{C}$ includes at least one infinite periodic group and is closed under taking subgroups, extensions, and Cartesian powers, there exists a prime number p (assumed to be fixed below) such that $\mathcal{C}$ contains a cyclic group of order p, the Cartesian product of an infinite number of such groups, and the class of all finite p-groups. Hence, the groups H_∞, A_∞, and B_∞ from Proposition 4.3 belong to $\mathcal{C}$. Let us define a graph of groups $\mathcal{G}(\Gamma)$ as follows.

If Γ has only one vertex, then we assign to it the group H_∞, while to each edge $e \in \mathcal{E}$ the same group H_∞ and the homomorphisms $\varphi_{+e} = \mathrm{id}_{H_\infty}$, $\varphi_{-e} = \alpha_\infty^{-1}\beta_\infty$, where α_∞ and β_∞ are the automorphisms from Proposition 4.3. Otherwise, we choose some edge f of Γ that is not a loop and associate the vertex $f(1)$ with the group A_∞, all other vertices with the group B_∞, and all the edges with the group H_∞ and its identity embeddings in A_∞ and B_∞.

It is easy to see that, in both cases and for any choice of the maximal tree $\mathcal{T}$ in Γ, all edge subgroups of the group $\pi_1(\mathcal{G}(\Gamma)) = \pi_1(\mathcal{G}(\Gamma), \mathcal{T})$ coincide, are normal in $\pi_1(\mathcal{G}(\Gamma))$, and are equal to H_∞ (we again denote this unique subgroup by H_∞). To prove Statement 3, let us fix an element $g \in \pi_1(\mathcal{G}(\Gamma)) \setminus \{1\}$ and find a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a $\mathcal{C}$-group taking g to a non-trivial element.

The quotient group $\pi_1(\mathcal{G}(\Gamma))/H_\infty$ is either a free group whose basis is the stable letters of $\pi_1(\mathcal{G}(\Gamma))$, or the (ordinary) free product of this group and finite p-groups, which belong to $\mathcal{C}$ and are isomorphic to A_∞/H_∞ or, what is the same, B_∞/H_∞. Hence, it is residually a $\mathcal{C}$-group by Proposition 3.3, and if $g \notin H_\infty$, then the natural homomorphism $\pi_1(\mathcal{G}(\Gamma)) \to \pi_1(\mathcal{G}(\Gamma))/H_\infty$ can be extended to the desired one.

Let $g \in H_\infty$. Consider the mapping of the generators of $\pi_1(\mathcal{G}(\Gamma))$ into the group P_∞ from Proposition 4.3 that acts on the generators of all vertex groups as the identity mapping and takes the symbols t_e either to the element $\alpha_\infty^{-1}\beta_\infty$ (if Γ has one vertex), or to 1 (otherwise). It is easy to see that this map defines a homomorphism, which is injective on H_∞. The group P_∞ is residually p-finite by Proposition 4.3, and the class $\mathcal{C}$ contains all finite p-groups. Hence, the constructed homomorphism can again be extended to the desired one.

Let us now turn to the proof of Statement 4 and show that there exists an element $\gamma \in \pi_1(\mathcal{G}(\Gamma))$ such that the conjugation by γ acts on H_∞ as the automorphism $\alpha_\infty^{-1}\beta_\infty$.

Indeed, if Γ has one vertex, then for some $e \in \mathcal{E}$, the element t_e can be taken as γ. Otherwise, $\gamma = \alpha_\infty^{-1}\beta_\infty$ or $\gamma = (t_f^{-1}\alpha_\infty t_f)^{-1}\beta_\infty$; it depends on whether or not $\mathcal{T}$ contains the edge f chosen above (here α_∞ and β_∞ are the elements of the groups A_∞ and B_∞ associated with the vertices $f(1)$ and $f(-1)$).

If σ is a homomorphism of $\pi_1(\mathcal{G}(\Gamma))$ onto a periodic group, then $(\gamma\sigma)^k = 1$ for some $k \geqslant 1$. Since, by Proposition 4.3, the automorphism $\alpha_\infty^{-1}\beta_\infty$ has an infinite order, there exists an element $h \in H_\infty$ satisfying the relations

$$h \neq h(\alpha_\infty^{-1}\beta_\infty)^k = \gamma^{-k}h\gamma^k.$$

Therefore,

$$1 \neq [h, \gamma^k] \in \ker\sigma \cap H_\infty = \ker\sigma \cap H_{\pm e}$$

for all $e \in \mathcal{E}$. □

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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