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Research Paper

On the residual nilpotence of generalized free products of groups

E.V. Sokolov

Ivanovo State University, Ermak st., 39, Ivanovo, 153025, Russia

A R T I C L E I N F O

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Let G be the generalized free product of two groups with an amalgamated subgroup. We propose an approach that allows one to use results on the residual p-finiteness of G for proving that this generalized free product is residually a finite nilpotent group or residually a finite metanilpotent group. This approach can be applied under most of the conditions on the amalgamated subgroup that allow the study of residual p-finiteness. Namely, we consider the cases where the amalgamated subgroup is a) periodic, b) locally cyclic, c) central in one of the free factors, d) normal in both free factors, or e) is a retract of one of the free factors. In each of these cases, we give certain necessary and sufficient conditions for G to be residually a) a finite nilpotent group, b) a finite metanilpotent group.

1. Introduction

Let $\mathcal{C}$ be a class of groups. Following [24], we say that a group X is *residually a $\mathcal{C}$-group* if, for any element $x \in X \setminus \{1\}$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$

E-mail address: ev-sokolov@yandex.ru.

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(a $\mathcal{C}$*-group*) such that $x\sigma \neq 1$. The terms "*residual finiteness*", "*residual nilpotence*", "*residual solvability*", and "*residual p-finiteness*" are used if $\mathcal{C}$ is the class of all finite groups, all nilpotent groups, all solvable groups, and finite p-groups for some prime p, respectively. In this article, we study the residual $\mathcal{C}$-ness of a generalized free product of two groups provided $\mathcal{C}$ is the class of finite nilpotent or finite metanilpotent groups.

Recall that a group is said to be *metanilpotent* if it is an extension of a nilpotent group by a nilpotent group. The definition of the construction of the generalized free product of two groups with an amalgamated subgroup can be found in Section 5. It should also be noted that a finitely generated group is residually nilpotent if and only if it is residually a finite nilpotent group (see Proposition 3.7 below).

Let us preface the description of the results of this article with a brief survey of known facts on the residual nilpotence of generalized free products. Since, for any choice of prime number p, every finite p-group is nilpotent, residual p-finiteness is a special case of residual nilpotence. Most of the known results on the residual nilpotence of generalized free products actually give necessary and/or sufficient conditions only for the residual p-finiteness of this construction. This is explained by the fact that the class of finite p-groups is closed under extensions. Thanks to this property, to prove the residual p-finiteness of a generalized free product, one can use very productive methods, which were originally proposed for studying the residual finiteness of this construction (primarily we mean here the so-called "filtration approach" by G. Baumslag [10]).

A criterion for the residual p-finiteness of a generalized free product of two finite groups is given in [25]. In the case of infinite free factors, residual p-finiteness is studied mainly under various additional restrictions imposed on the amalgamated subgroup. For example, in [1,3,6], a number of theorems are proved that generalize the criterion from [25] and give conditions for the residual p-finiteness of a generalized free product with a finite amalgamation. Much effort is put into studying the residual p-finiteness of free products with a cyclic amalgamated subgroup [2,5,18,22,31,32,56] and with an amalgamated subgroup that is normal in every free factor [4,30,45,50,53,54]. A free product is also considered whose amalgamated subgroup lies in the center of at least one free factor [51] or is a retract of this factor [16,49]. In particular, the following criteria are proved:

– a criterion for the residual p-finiteness of a generalized free product of two isomorphic groups [31, Corollary 3.5];

– a criterion for the residual p-finiteness of a generalized free product whose amalgamated subgroup is a retract of each free factor [16, Corollary 1.2];

– criteria for the residual p-finiteness of a generalized free product with a cyclic amalgamation of a) two free groups [2, Theorem 1], [32, Theorem 3.5]; b) two nilpotent groups of finite rank [5, Theorem 2];

– criteria for the residual p-finiteness of a generalized free product such that its amalgamated subgroup is normal in every free factor and at least one of the following additional conditions holds: a) each free factor is virtually a solvable group of finite rank [4, Theorem 2]; b) the amalgamated subgroup is finite [53, Theorem 3]; c) one of the free factors is a finitely generated nilpotent group, and the amalgamated subgroup lies in its center [30, Theorem 4.9] or is cyclic [53, Theorem 5].

Recall that a group is said to have *finite rank* r if any of its finitely generated subgroups can be generated by at most r elements.

If $\mathcal{C}$ is a class of nilpotent groups that is not closed under taking extensions, there are no general methods for studying the residual $\mathcal{C}$-ness of generalized free products. As a consequence, very few results are known in this case. Only the following is proved:

– a criterion for the residual nilpotence of a generalized free product of two finite groups [55, Theorem 6] and its generalization to the case of a free product of two finitely generated nilpotent groups with a finite amalgamation [27, Theorem 4];

– a criterion for the residual nilpotence of a generalized free product of two nilpotent groups with an amalgamated subgroup lying in the center of each free factor [42, Theorem 1];

– three criteria for the residual nilpotence of a generalized free product of two finitely generated nilpotent groups; their description is given at the end of the next section [27, Theorems 5, 7 and Corollary of Theorem 6];

– a criterion for a generalized free product of two nilpotent groups to be residually a finite nilpotent group, which holds provided the amalgamated subgroup is central in one and normal in the other free factor [42, Theorem 2];

– several sufficient conditions for a generalized free product with a cyclic amalgamation to be residually a torsion-free nilpotent group; the cases are considered when the free factors are free groups [12–14,33] or one of these factors is a free group and the other is a free abelian group [9].

It is proved also that, with a suitable choice of a prime p, the residual nilpotence is equivalent to the residual p-finiteness for the following constructions:

– a generalized free product of two free groups with a cyclic amalgamation [2, Theorem 3];

– a generalized free product of two finitely generated torsion-free nilpotent groups whose amalgamated subgroup is cyclic or lies in the center of each free factor [27, Corollary of Theorem 6].

This paper significantly complements the above list. It proposes a general approach that allows one to use results on the residual p-finiteness of a generalized free product G to study the residual $\mathcal{C}$-ness of G, where $\mathcal{C}$ is the class of finite nilpotent or finite metanilpotent groups. We show that this approach can be applied under each of the restrictions on the amalgamated subgroup, which are mentioned above and allow the study of residual p-finiteness. The results obtained in this way are formulated in the next section, while here we give a number of concepts and notations used throughout the paper.

Suppose that $\mathfrak{P}$ is a set of prime numbers, X is a group, and Y is a subgroup of X. Recall that an integer is said to be a $\mathfrak{P}$-*number* if all its prime divisors belong to $\mathfrak{P}$; a periodic group is said to be a $\mathfrak{P}$-*group* if the orders of all its elements are $\mathfrak{P}$-numbers. We denote by $\mathfrak{P}'$ the set of all prime numbers that do not belong to $\mathfrak{P}$.

The subgroup Y is called $\mathfrak{P}'$*-isolated* in X if, for any element $x \in X$ and for any number $q \in \mathfrak{P}'$, it follows from the inclusion $x^q \in Y$ that $x \in Y$. If the trivial subgroup of X is $\mathfrak{P}'$-isolated, then X is said to be a $\mathfrak{P}'$*-torsion-free group*. It is easy to see that the intersection of any number of $\mathfrak{P}'$-isolated subgroups is again a $\mathfrak{P}'$-isolated subgroup and therefore one can define the smallest $\mathfrak{P}'$-isolated subgroup containing the subgroup Y. It is called the $\mathfrak{P}'$*-isolator* of Y in X and is further denoted by $\mathfrak{P}'\text{-}\mathfrak{Is}(X, Y)$. Obviously, $\mathfrak{P}'\text{-}\mathfrak{Is}(X, Y)$ contains the subset $\mathfrak{P}'\text{-}\mathfrak{Rt}(X, Y)$ of X such that $x \in \mathfrak{P}'\text{-}\mathfrak{Rt}(X, Y)$ if and only if $x^n \in Y$ for some $\mathfrak{P}'$-number n. It is clear that the equality $\mathfrak{P}'\text{-}\mathfrak{Is}(X, Y) = \mathfrak{P}'\text{-}\mathfrak{Rt}(X, Y)$ holds if and only if the set $\mathfrak{P}'\text{-}\mathfrak{Rt}(X, Y)$ is a subgroup.

Following [38], we say that the subgroup Y is *separable in X by the class of groups $\mathcal{C}$* (or, more briefly, is $\mathcal{C}$*-separable in X*) if, for any element $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ such that $x\sigma \notin Y\sigma$. As can be seen from this definition, the residual $\mathcal{C}$-ness of X is equivalent to the $\mathcal{C}$-separability of its trivial subgroup. It is also easy to prove that if the subgroup Y is separable in X by the class of finite $\mathfrak{P}$-groups, then it is $\mathfrak{P}'$-isolated in X (see Proposition 3.2 below).

Let us call an abelian group $\mathfrak{P}$*-bounded* if, in each of its quotient groups, a primary component of the periodic part is finite whenever it corresponds to a number from $\mathfrak{P}$. We say also that a nilpotent group is $\mathfrak{P}$*-bounded* if it has a finite central series with $\mathfrak{P}$-bounded abelian factors. The classes of $\mathfrak{P}$-bounded abelian and $\mathfrak{P}$-bounded nilpotent groups are further denoted by $\mathcal{BA}_{\mathfrak{P}}$ and $\mathcal{BN}_{\mathfrak{P}}$, respectively. Obviously, if $\mathfrak{S} \subseteq \mathfrak{P}$, then $\mathcal{BA}_{\mathfrak{P}} \subseteq \mathcal{BA}_{\mathfrak{S}}$ and $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_{\mathfrak{S}}$. It is also easy to see that a finitely generated nilpotent group is $\mathfrak{P}$-bounded for any choice of the set $\mathfrak{P}$.

Everywhere below let Φ, $\mathcal{F}_{\mathfrak{P}}$, and $\mathcal{FN}_{\mathfrak{P}}$ stand for the classes of free groups, finite $\mathfrak{P}$-groups, and finite nilpotent $\mathfrak{P}$-groups, respectively. If $\mathfrak{P} = \{p\}$, we simply write p instead of $\{p\}$ (for example: p-number, $\mathcal{BN}_p$, $p'\text{-}\mathfrak{Is}(X, Y)$). We also use a number of standard notations:

$\lvert X\rvert$	the order of a group X;
$[x, y]$	the commutator of elements x and y, which is assumed to be equal to the product $x^{-1}y^{-1}xy$;
$\mathrm{sgp}\{S\}$	the subgroup generated by a set S;
X'	the commutator subgroup of X, i.e., $\mathrm{sgp}\{[x, y] \mid x, y \in X\}$;
X^n	$\mathrm{sgp}\{x^n \mid x \in X\}$;
$\ker\sigma$	the kernel of a homomorphism σ;
$\mathcal{C} \cdot \mathcal{D}$	the class of groups consisting of all possible extensions of a $\mathcal{C}$-group by a $\mathcal{D}$-group, where $\mathcal{C}$ and $\mathcal{D}$ are some classes of groups.

2. New results

Let further the expression $G = \langle A * B;\ H\rangle$ mean that G is the generalized free product of groups A and B with an amalgamated subgroup H (as noted above, the definition and necessary properties of this construction are given in Section 5). We say that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ *satisfy Condition* $(*)$ if

(i) $A \neq H \neq B$;

(ii) $\mathfrak{P}$ is a non-empty set;

(iii) A and B are residually $\mathcal{FN}_{\mathfrak{P}}$-groups;

(iv) there exist homomorphisms of the groups A and B that map them onto $\mathcal{BN}_{\mathfrak{P}}$-groups and act injectively on H.

Almost all the main results of this paper deal with generalized free products satisfying (∗). Parts (i)—(iii) of this condition look quite natural because a) if G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then A and B have the same property; b) if $\mathfrak{P} = \varnothing$, only the trivial group is residually an $\mathcal{FN}_{\mathfrak{P}}$-group; c) if $A = H$ or $B = H$, then $G = B$ or $G = A$ and it follows from (iii) that G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group. The more artificial restriction (iv) is necessary to obtain the results mentioned above via Theorem 7. This theorem is given in Section 8 and allows one to prove the residual nilpotence of G using known facts on the residual p-finiteness of this group.

It is easy to see that if H is finite, then (iii) implies (iv). In the case of the infinite subgroup H, homomorphisms satisfying (iv) certainly exist if A and B are residually torsion-free $\mathcal{BN}_{\mathfrak{P}}$-groups and H is of finite Hirsch–Zaitsev rank, i.e., it has a finite subnormal series whose each factor is a periodic or infinite cyclic group (see Proposition 3.5 below). The following groups are examples of residually torsion-free $\mathcal{BN}_{\mathfrak{P}}$-groups:

– free [36], parafree [11], limit [43], and free polynilpotent [23] groups;

– fundamental groups of two-dimensional orientable closed manifolds [9];

– pure braid groups of various types [8,15,20,39,40];

– right-angled Artin groups [19] and their Torelli groups [52];

– Hydra groups and some other groups with one defining relation, including the cyclically pinched one-relator groups mentioned in the previous section [12–14,33];

– all restricted and unrestricted direct products of the groups listed above.

The following theorem is the first of the main results of this article.

Theorem 1. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy* (∗) *and at least one of the next additional conditions*:

(α) *H is locally cyclic*;

(β) *H lies in the center of A or B*;

(γ) *H is a retract of A or B.*

Then the following statements hold.

1. *If H is $\mathcal{F}_q$-separable in A and B for some $q \in \mathfrak{P}$, then G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group.*

2. *If H is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and B, then G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and residually an $\mathcal{F}_p \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each prime p. In particular, it is residually a finite metanilpotent $\mathfrak{P}$-group.*

If X is a group and Y is its normal subgroup, then the restriction of any inner automorphism of X onto the subgroup Y is an automorphism of the latter. The set of all such automorphisms forms a subgroup of the group $\operatorname{Aut} Y$, which we further denote by $\mathrm{Aut}_X(Y)$. If $G = \langle A * B;\ H\rangle$ and the subgroup H is normal in A and B, then it is also normal in G and this allows us to define the group $\mathrm{Aut}_G(H)$. It turns out that the properties of the latter can be used to describe the conditions for the residual $\mathcal{C}$-ness of G; this approach was first used in [25] and then repeatedly in other works (see, for example, [53,54]). Let us note also that if H is normal in G, then, for each prime number p, the subgroup H^pH' has the same property and therefore the groups

$$\mathfrak{A}(p) = \mathrm{Aut}_{A/H^pH'}(H/H^pH'), \quad \mathfrak{B}(p) = \mathrm{Aut}_{B/H^pH'}(H/H^pH'), \quad \text{and}$$
$$\mathfrak{G}(p) = \mathrm{Aut}_{G/H^pH'}(H/H^pH')$$

are defined.

Theorem 2. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that H is normal in A and B and, for each $p \in \mathfrak{P}$, at least one of the next additional conditions holds:*
(α) $\mathfrak{G}(p)$ is a p-group;
(β) $\mathfrak{G}(p)$ is abelian;
(γ) $\mathfrak{G}(p) = \mathfrak{A}(p)$ or $\mathfrak{G}(p) = \mathfrak{B}(p)$.
Then the following statements hold.
1. *If H is $\mathcal{F}_q$-separable in A and B for some $q \in \mathfrak{P}$, then G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group.*
2. *If G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then H is q'-isolated in A and B for some $q \in \mathfrak{P}$.*
3. *The group G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and residually an $\mathcal{F}_p \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each prime p if and only if H is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and B.*

It should be noted that Theorem 2 is a special case of Theorem 6. The formulation of the latter requires the preliminary proof of some auxiliary facts and is therefore given in Section 5. We note also that if the subgroup H/H^pH' lies in the center, say, of the group A/H^pH', then $\mathfrak{A}(p) = 1$ and $\mathfrak{G}(p) = \mathfrak{B}(p)$ because $\mathfrak{G}(p)$ is obviously generated by $\mathfrak{A}(p)$ and $\mathfrak{B}(p)$. If the indicated subgroup is locally cyclic, then its automorphism group is abelian (see, for example, [21, §113, Exercise 4]). Thus, the next corollary follows from Theorem 2.

Corollary 1. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that H is normal in A and B and, for each $p \in \mathfrak{P}$, the subgroup H/H^pH' is locally cyclic or lies in the center of at least one of the groups A/H^pH', B/H^pH'. Then Statements 1—3 of Theorem 2 hold.*

Theorem 3. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that H is periodic and, for every prime p, the symbol $H(p)$ denotes the subgroup p-$\mathfrak{Is}(H, 1)$. Then H is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and B, and the following statements hold.*

1. *The group G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group if H is $\mathcal{F}_q$-separable in A and B for some $q \in \mathfrak{P}$ and, for any $p \in \mathfrak{P}$, there exist sequences of subgroups*

$$1 = Q_0 \leqslant Q_1 \leqslant \ldots \leqslant Q_n = H(p), \quad R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A, \quad \textit{and}$$
$$S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B$$

such that, for all $i \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, n-1\}$,
- *R_i is a normal subgroup of finite p-index of the group A;*
- *S_i is a normal subgroup of finite p-index of the group B;*
- *$R_i \cap H(p) = Q_i = S_i \cap H(p)$;*
- *$|Q_{j+1}/Q_j| = p$.*

2. *If G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then H is q'-isolated in A and B for some $q \in \mathfrak{P}$ and, for any $p \in \mathfrak{P}$, there exist the sequences of subgroups described in Statement 1 of this theorem.*

3. *The group G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and residually an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each prime q if and only if, for any $p \in \mathfrak{P}$, there exist the sequences of subgroups described in Statement 1 of this theorem.*

Let τA and τB be the sets of all elements of finite order of the groups A and B, respectively. If the indicated groups have homomorphisms onto $\mathcal{BN}_{\mathfrak{P}}$-groups acting injectively not only on H, but also on the subgroups $\mathrm{sgp}\{\tau A\}$ and $\mathrm{sgp}\{\tau B\}$, then Theorem 3 admits the next equivalent formulation generalizing Theorem 4 from [6].

Theorem 4. *Suppose that the group $G = \langle A*B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that H is periodic and there exist homomorphisms of the groups A and B that map them onto $\mathcal{BN}_{\mathfrak{P}}$-groups and act injectively on the subgroups $\mathrm{sgp}\{\tau A\}$ and $\mathrm{sgp}\{\tau B\}$. Then, for each $p \in \mathfrak{P}$, the sets $A(p) = p\text{-}\mathfrak{Rt}(A, 1)$ and $B(p) = p\text{-}\mathfrak{Rt}(B, 1)$ are finite normal subgroups of A and B, respectively, H is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and B, and the following statements hold.*

1. *The group G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group if H is $\mathcal{F}_q$-separable in A and B for some $q \in \mathfrak{P}$ and, for any $p \in \mathfrak{P}$, the groups $A(p)$ and $B(p)$ have normal series*

$$1 = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_k = A(p) \quad \textit{and} \quad 1 = B_0 \leqslant B_1 \leqslant \ldots \leqslant B_m = B(p)$$

with factors of order p such that, for all $i \in \{0, 1, \ldots, k\}$, $j \in \{0, 1, \ldots, m\}$,
- *A_i is a normal subgroup of A;*
- *B_j is a normal subgroup of B;*
- *$\{A_i \cap H \mid 0 \leqslant i \leqslant k\} = \{B_j \cap H \mid 0 \leqslant j \leqslant m\}$.*

2. *If G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then H is q'-isolated in A and B for some $q \in \mathfrak{P}$ and, for any $p \in \mathfrak{P}$, there exist normal series described in Statement 1 of this theorem.*

3. *The group G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and residually an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each prime q if and only if, for any $p \in \mathfrak{P}$, the groups $A(p)$ and $B(p)$ have normal series described in Statement 1.*

Let us now turn to the question on the conditions that are necessary for a generalized free product to be residually a nilpotent or metanilpotent group.

Theorem 5. *Suppose that $G = \langle A * B;\ H\rangle$, $\mathfrak{P}$ is a non-empty set of primes, and H is nilpotent. Suppose also that*

(α) H does not coincide with its normalizer in A or is properly contained in some subgroup of A satisfying a non-trivial identity;

(β) H does not coincide with its normalizer in B or is properly contained in some subgroup of B satisfying a non-trivial identity.

Then the following statements hold.

1. *If G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then H is q'-isolated in A and B for some $q \in \mathfrak{P}$.*
2. *If G is residually a $\mathcal{C}$-group for some subclass $\mathcal{C}$ of the class $\mathcal{F}_{\mathfrak{P}}$ and $\mathcal{C}$ is closed under taking subgroups, then H is $\mathcal{C}$-separable and therefore $\mathfrak{P}'$-isolated in A and B.*

The next corollary is obtained by combining Theorems 1, 5 and Proposition 9.1 given below.

Corollary 2. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy ($*$), Conditions (α) and (β) of Theorem 5, and at least one of Conditions (α)—(γ) of Theorem 1. Then Statements 1—3 of Theorem 2 hold.*

The following example shows, in particular, that, in Theorem 5, none of Conditions (α) and (β) can be omitted.

Example 1. Suppose that $\mathfrak{P}$ is a non-empty set of prime numbers, which does not coincide with the set of all primes, and $G = \langle a_1, a_2, b;\ a_2 = b^n\rangle$, where $n \in \mathfrak{P}'$. Then G is the generalized free product of the non-abelian free group $A = \mathrm{sgp}\{a_1, a_2\}$ and the infinite cyclic group $B = \mathrm{sgp}\{b\}$ with the amalgamated cyclic subgroup $H = \mathrm{sgp}\{b^n\}$, which lies in the center of B and is a retract of A. At the same time, the elements a_1 and b freely generate the group G and therefore the latter is residually an $\mathcal{FN}_{\mathfrak{P}}$-group (see Proposition 3.7 below). It is also obvious that the retraction of A and the identity homomorphism of B act injectively on H and map A and B onto infinite cyclic groups, which belong to the class $\mathcal{BN}_{\mathfrak{P}}$. Thus, G satisfies all the conditions of Theorem 1. At the same time, H is neither $\mathfrak{P}'$-isolated nor $\mathcal{FN}_{\mathfrak{P}}$-separable in B. Also, it is neither q'-isolated nor $\mathcal{F}_q$-separable in this group for any $q \in \mathfrak{P}$.

Example 2. Let

$$G = \big\langle a, b, c;\ a^9 = [a^3, b] = [a^3, c] = c^{-1}bcb = 1\big\rangle$$

and $\mathfrak{P} = \{2, 3\}$. It is obvious that

1) G is the generalized free product of the groups $A = \mathrm{sgp}\{a\}$ and $B = \mathrm{sgp}\{a^3, b, c\}$ with the subgroup $H = \mathrm{sgp}\{a^3\}$ amalgamated;

2) A is a finite cyclic group of order 9 and therefore belongs to the class $\mathcal{F}_3 \subseteq \mathcal{BN}_{\mathfrak{P}}$;

3) B can be decomposed into the direct product of the subgroups H and $C = \operatorname{sgp}\{b, c\}$; moreover, C is the non-abelian extension of the infinite cyclic group $\operatorname{sgp}\{b\}$ by the infinite cyclic group $\operatorname{sgp}\{c\}$;

4) the finite cyclic group H lies in the center of G and is a retract of B; the identity homomorphism of A and the retraction of B act injectively on H and map A and B onto groups from the class $\mathcal{F}_3 \subseteq \mathcal{BN}_{\mathfrak{P}}$;

5) if we define the group $\mathfrak{G}(p)$ in the same way as in Theorem 2, then $\mathfrak{G}(3) \cong \operatorname{Aut}_G(H) = 1$ and $\mathfrak{G}(2) = \operatorname{Aut}_{G/H}(1) = 1$;

6) the following sequences satisfy the conditions of Theorem 3: $1 = Q_0 = H(p)$, $R_0 = A$, and $S_0 = B$ if $p = 2$; $1 = Q_0 \leqslant Q_1 = H(p) = H$, $1 = R_0 \leqslant R_1 = A$, and $C = S_0 \leqslant S_1 = B$ if $p = 3$;

7) the following normal series satisfy the conditions of Theorem 4: $1 = A_0 = A(p)$ and $1 = B_0 = B(p)$ if $p = 2$; $1 = A_0 \leqslant H \leqslant A = A(p)$ and $1 = B_0 \leqslant H = B(p)$ if $p = 3$.

It is well known (see, for example, [41, Theorem 2]) that C is residually an $\mathcal{F}_2$-group. Therefore, B is residually an $\mathcal{FN}_{\mathfrak{P}}$-group and G satisfies all the conditions of Theorems 1—4 and Corollaries 1, 2. Moreover, since the group $C \cong B/H$ is torsion-free, H is isolated in B and therefore $3'$-isolated in A and B. However, G is not residually an $\mathcal{FN}_{\mathfrak{P}}$-group by Proposition 6.2 given below.

Examples 1 and 2 imply the following.

1. In Theorem 1, the $\mathcal{FN}_{\mathfrak{P}}$-separability of H in A and B is, in the general case, not necessary for G to be residually an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group, where $q \in \mathfrak{P}$; the fact that G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group does not necessarily mean that H is $\mathcal{F}_q$-separable or even q'-isolated in A and B for some $q \in \mathfrak{P}$.

2. In Theorems 2—4 and Corollaries 1, 2, the property of H to be q'-isolated in A and B for some $q \in \mathfrak{P}$ is necessary, but not sufficient for G to be residually an $\mathcal{FN}_{\mathfrak{P}}$-group. The question on the necessity of the $\mathcal{F}_q$-separability of H remains open.

Let us note that $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_q$ for any $q \in \mathfrak{P}$ and therefore, by Proposition 4.4 given below, a subgroup of a $\mathcal{BN}_{\mathfrak{P}}$-group X is $\mathcal{F}_q$-separable ($\mathcal{FN}_{\mathfrak{P}}$-separable) in this group if and only if it is q'-isolated (respectively, $\mathfrak{P}'$-isolated) in X. Since every $\mathcal{BN}_{\mathfrak{P}}$-group also satisfies a non-trivial identity, the next corollary follows from Theorems 1—5.

Corollary 3. *Suppose that $G = \langle A * B;\ H\rangle$, $\mathfrak{P}$ is a non-empty set of primes, A and B are $\mathfrak{P}'$-torsion-free $\mathcal{BN}_{\mathfrak{P}}$-groups, and $A \neq H \neq B$. Then the following statements hold.*

1. If at least one of the conditions (α)—(γ) of Theorem 1 is satisfied, then G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group (an $\mathcal{FN}_{\mathfrak{P}} \cdot \mathcal{FN}_{\mathfrak{P}}$-group) if and only if H is q'-isolated in A and B for some $q \in \mathfrak{P}$ (respectively, H is $\mathfrak{P}'$-isolated in these groups).

2. If H is normal in A and B and, for each $p \in \mathfrak{P}$, at least one of the conditions (α)—(γ) of Theorem 2 is satisfied, then G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group (an $\mathcal{FN}_{\mathfrak{P}} \cdot \mathcal{FN}_{\mathfrak{P}}$-group) if and only if H is q'-isolated in A and B for some $q \in \mathfrak{P}$ (respectively, H is $\mathfrak{P}'$-isolated in these groups).

3. *If H is periodic, then the following statements are equivalent*:

(α) *G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group*;

(β) *for some $q \in \mathfrak{P}$, H is q'-isolated in A and B, and, for any $p \in \mathfrak{P}$, there exist sequences of subgroups described in Statement* 1 *of Theorem* 3;

(γ) *for some $q \in \mathfrak{P}$, H is q'-isolated in A and B, and, for any $p \in \mathfrak{P}$, there exist normal series described in Statement* 1 *of Theorem* 4.

When the groups A and B are finitely generated, the group $G = \langle A * B;\ H\rangle$ is residually nilpotent if and only if it is residually a finite nilpotent group. Therefore, Statement 1 of Corollary 3 generalizes

– the criteria for the residual nilpotence of generalized free products of a) two finitely generated abelian groups [27, Theorem 5]; b) two finitely generated nilpotent groups with a cyclic amalgamated subgroup [27, Theorem 7]; c) two finitely generated torsion-free nilpotent groups with an amalgamated subgroup lying in the center of each free factor [27, Corollary of Theorem 6];

– the criteria for the residual p-finiteness of generalized free products of a) two finitely generated abelian groups [30, Corollary 4.8]; b) two finitely generated nilpotent groups with a cyclic amalgamated subgroup [27, Theorem 6] (for the case of torsion-free factors, this result was obtained earlier in [32, Theorem 4.4]); c) two finitely generated nilpotent groups with an amalgamated subgroup lying in the center of each free factor [30, Corollary 4.7].

3. Some auxiliary statements

Everywhere below, if $\mathcal{C}$ is an arbitrary class of groups and X is a group, then $\mathcal{C}^*(X)$ denotes the family of normal subgroups of X defined as follows: $Y \in \mathcal{C}^*(X)$ if and only if $X/Y \in \mathcal{C}$.

Proposition 3.1. *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors. Suppose also that X is a group, Y and Z are subgroups of X. Then the following statements hold.*

1. *If $Y \in \mathcal{C}^*(X)$, then $Y \cap Z \in \mathcal{C}^*(Z)$.*
2. *If $Y, Z \in \mathcal{C}^*(X)$, then $Y \cap Z \in \mathcal{C}^*(X)$.*
3. *If X is residually a $\mathcal{C}$-group, then, for any finite set $S \subseteq X \setminus \{1\}$, there exists a subgroup $T \in \mathcal{C}^*(X)$ such that $T \cap S = \varnothing$.*

Proof. Indeed, if $Y \in \mathcal{C}^*(X)$, then $Z/Y \cap Z \cong ZY/Y \leqslant X/Y \in \mathcal{C}$ and since $\mathcal{C}$ is closed under taking subgroups, $Z/Y \cap Z \in \mathcal{C}$. If $Y, Z \in \mathcal{C}^*(X)$, then, by Remak's theorem (see, for example, [28, Theorem 4.3.9]), the quotient group $X/Y \cap Z$ can be embedded into the direct product of the groups X/Y and X/Z. Hence, this group belongs to $\mathcal{C}$ because the latter is closed under taking subgroups and direct products. If X is residually a $\mathcal{C}$-group and $S \subseteq X \setminus \{1\}$, then, for each element $s \in S$, there exists a subgroup $T_s \in$

$\mathcal{C}^*(X)$ such that $s \notin T_s$. Since S is finite, it follows from Statement 2 that the subgroup $T = \bigcap_{s \in S} T_s$ is desired.

Proposition 3.2. [49, Proposition 5] *Suppose that $\mathcal{C}$ is a class of groups consisting only of periodic groups, X is a group, and Y is a subgroup of X. Suppose also that $\mathfrak{P}(\mathcal{C})$ is the set of primes defined as follows: $p \in \mathfrak{P}(\mathcal{C})$ if and only if p divides the order of some element of a $\mathcal{C}$-group. If the subgroup Y is $\mathcal{C}$-separable in X, then it is $\mathfrak{P}(\mathcal{C})'$-isolated in this group.*

Proposition 3.3. [54, Proposition 3] *Suppose that $\mathcal{C}$ is a class of groups closed under taking quotient groups, X is a group, and Y is a normal subgroup of X. The quotient group X/Y is residually a $\mathcal{C}$-group if and only if Y is $\mathcal{C}$-separable in X.*

Proposition 3.4. [54, Proposition 4] *Suppose that $\mathcal{C}$ is a class of groups closed under taking subgroups, quotient groups, and direct products of a finite number of factors. Suppose also that X is a group and Y is a finite normal subgroup of X. If X is residually a $\mathcal{C}$-group, then $\mathrm{Aut}_X(Y) \in \mathcal{C}$.*

Proposition 3.5. [54, Proposition 18] *Suppose that $\mathcal{C}$ is a class of torsion-free groups closed under taking subgroups and direct products of a finite number of factors. Suppose also that X is a group and Y is a subgroup of X. If X is residually a $\mathcal{C}$-group and Y is of finite Hirsch–Zaitsev rank, then there exists a subgroup $Z \in \mathcal{C}^*(X)$ such that $Y \cap Z = 1$.*

For the convenience of the reader, we give the next proposition along with its proof because the paper containing it is actually unavailable.

Proposition 3.6. [57, Lemma 2.4] *Suppose that p is a prime number and X is a finite p-group. Suppose also that α is an automorphism of X and δ is the automorphism of the quotient group X/X^pX' induced by α. Then the order of α is a p-number if the order of δ has the same property.*

Proof. Let Γ_i, q, and c denote the i-th member of the lower central series, the order, and the nilpotency class of X, respectively. Let also $\overline{\alpha}$ be the automorphism of the quotient group X/X' induced by α. Using induction on c, we first show that if the order of $\overline{\alpha}$ is a p-number, then the order of α is also a p-number. Since this is obvious for $c = 1$, we can further assume that $c > 1$.

Let Y, β, and $\bar{\beta}$ stand for the quotient group X/Γ_c and the automorphisms of the groups Y and $Y/Y' = (X/\Gamma_c)/(X'/\Gamma_c)$ induced by α and β, respectively. It is easy to see that if $\sigma\colon Y/Y' \to X/X'$ is the isomorphism defined by the rule $((x\Gamma_c)Y')\sigma = xX'$, then $\bar{\beta} = \sigma\overline{\alpha}\sigma^{-1}$. The last relation means that the order of $\bar{\beta}$ is equal to the order of $\overline{\alpha}$ and therefore is a p-number. By the inductive hypothesis applied to Y, the order r of β is also a p-number. It follows from the definition of β that if $x \in X$ and $y \in \Gamma_{c-1}$, then $x\alpha^r = xw_1$ and $y\alpha^r = yw_2$ for suitable elements $w_1, w_2 \in \Gamma_c$. Since Γ_c lies in the center of X, we have $[y, x]\alpha^r = [y, x][w_2, w_1] = [y, x]$. Thus, the automorphism α^r acts identically on Γ_c and therefore $x\alpha^{rq} = xw_1^q = x$ (recall that q denotes the order of X). Since x is chosen arbitrarily, it follows that $\alpha^{rq} = 1$, as required.

Now it remains to prove that if the order of the automorphism δ defined above is a p-number, then the order of $\overline{\alpha}$ has the same property. This is obvious if the quotient group $Z = X/X'$ is trivial. Thus, we can assume further that $Z \neq 1$.

Let $\overline{\gamma}$ be the automorphism of the quotient group Z/Z^p induced by the automorphism $\gamma = \overline{\alpha}$. As above, it is easy to show that the order of δ is equal to the order of $\overline{\gamma}$. Let us fix a decomposition of Z into the direct product of non-trivial cyclic groups with generators $z_1, z_2, \ldots, z_n$. Since $Z \neq 1$, the relation $n \geqslant 1$ holds, and the automorphism γ can be given by the integer matrix $\Theta = \{\theta_{ij}\}_{i,j=1}^n$ defined by the equalities $z_i\gamma = z_1^{\theta_{i1}} z_2^{\theta_{i2}} \ldots z_n^{\theta_{in}}$, where $1 \leqslant i \leqslant n$. It is clear that the group Z/Z^p, the set $\{z_1, z_2, \ldots, z_n\}$, and the mapping $\overline{\gamma}$ can be viewed as a linear space over the field $\mathbb{Z}_p$, a basis, and a linear operator of this space, respectively. If the elements of Θ are considered representatives of residue classes modulo p, then the matrix of $\overline{\gamma}$ in the indicated basis coincides with Θ and therefore $\Theta^s = 1$ for some p-number s. It easily follows that if the elements of Θ are considered representatives of residue classes modulo p^k for some $k \geqslant 1$, then $\Theta^{sp^{k-1}} = 1$. Since the orders of the elements $z_1, z_2, \ldots, z_n$ divide the order q of X, the last equality means that $\gamma^{sq} = 1$, as required.

Proposition 3.7. *The following statements hold.*

1. *Any free group is residually a finitely generated torsion-free nilpotent group* [36,37].
2. *Any polycyclic group is residually finite* [26].
3. *Any finitely generated torsion-free nilpotent group is residually an $\mathcal{F}_p$-group for every prime p* [23].
4. *A group is residually an $\mathcal{FN}_{\mathfrak{P}}$-group if and only if it is residually a $\mathcal{C}$-group, where $\mathcal{C} = \bigcup_{p \in \mathfrak{P}} \mathcal{F}_p$.*
5. *Any free group is residually an $\mathcal{F}_p$-group for each prime number p and is residually an $\mathcal{FN}_{\mathfrak{P}}$-group for each non-empty set of primes $\mathfrak{P}$.*
6. *If a finitely generated group is residually nilpotent, then it is residually a $\mathcal{D}$-group, where $\mathcal{D}$ is the union of the classes $\mathcal{F}_p$ over all primes p.*

Proof. It is well known that any finite nilpotent group can be decomposed into the direct product of its Sylow subgroups (see, for example, Proposition 4.1 below). Therefore, every $\mathcal{FN}_{\mathfrak{P}}$-group is residually a $\mathcal{C}$-group, and Statement 4 holds. Statement 5 follows from Statements 1, 3, and 4. Any nilpotent image of a finitely generated group is a finitely generated nilpotent group, which is polycyclic and therefore residually finite by Statement 2. In fact, this image is residually a finite nilpotent group since the class of nilpotent groups is closed under taking subgroups. Thus, Statement 6 follows from Statement 4.

Proposition 3.8. *If $\mathcal{C}$ is a class of groups consisting only of finite groups, then any extension of a free group by a $\mathcal{C}$-group is residually an $\mathcal{F}_p \cdot \mathcal{C}$-group for each prime number p.*

Proof. Suppose that X is an extension of a free group Y by a $\mathcal{C}$-group, $x \in X \setminus \{1\}$, and p is a prime number. To prove the proposition it is sufficient to find a homomorphism of X onto a group from $\mathcal{F}_p \cdot \mathcal{C}$ taking x to a non-trivial element.

If $x \notin Y$, then the natural homomorphism $X \to X/Y$ is the desired one. Therefore, we can further assume that $x \in Y$. By Proposition 3.7, Y is residually an $\mathcal{F}_p$-group. Hence, there exists a subgroup $M \in \mathcal{F}_p^*(Y)$ such that $x \notin M$. If S is a set of representatives for all cosets of Y in X and $N = \bigcap_{s \in S} s^{-1}Ms$, then N is a normal subgroup of X. Since $X/Y \in \mathcal{C}$ and $\mathcal{C}$ consists of finite groups, the set S is also finite. It is easy to see that, for each $s \in S$, the quotient group $Y/s^{-1}Ms$ is isomorphic to the $\mathcal{F}_p$-group Y/M. Therefore, $Y/N \in \mathcal{F}_p$ by Proposition 3.1. Since $N \leqslant M$ and $x \notin M$, it follows that the natural homomorphism $X \to X/N$ is desired.

4. Some properties of nilpotent groups

Proposition 4.1. [17, Theorem 5.3, Lemma 5.5] *Suppose that $\mathfrak{P}$ is a set of primes, X is a locally nilpotent group, and Y is a subgroup of X. Then $\mathfrak{P}'$-$\mathfrak{Is}(X, Y) = \mathfrak{P}'$-$\mathfrak{Rt}(X, Y)$ and, if Y is normal in X, then $\mathfrak{P}'$-$\mathfrak{Is}(X, Y)$ has the same property. In particular, the set of all elements of finite order of X is a subgroup, which can be decomposed into the direct product of normal (in X) subgroups p-$\mathfrak{Rt}(X, 1)$ over all prime numbers p.*

Suppose that $\mathcal{C}$ is a class of groups, X is a group, and Y is a subgroup of X. Let us say that X is *$\mathcal{C}$-regular* (*$\mathcal{C}$-quasi-regular*) with respect to Y if, for any subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y = M$ (respectively $N \cap Y \leqslant M$). The next three propositions combine special cases of Propositions 5.2, 6.3 and Theorems 2.2, 2.4 from [48], which can be obtained by replacing the class $\mathcal{C}$ that appears in the formulations of these propositions and theorems with $\mathcal{F}_{\mathfrak{P}}$.

Proposition 4.2. *For any non-empty set of primes $\mathfrak{P}$, the following statements hold.*

1. *The classes $\mathcal{BA}_{\mathfrak{P}}$ and $\mathcal{BN}_{\mathfrak{P}}$ are closed under taking subgroups, quotient groups, and direct products of a finite number of factors.*
2. *Every abelian $\mathcal{BN}_{\mathfrak{P}}$-group belongs to the class $\mathcal{BA}_{\mathfrak{P}}$.*

Proposition 4.3. *Suppose that $\mathfrak{P}$ is a non-empty set of primes, X is a group, and Y is a subgroup of X. If there exists a homomorphism of X onto a $\mathcal{BN}_{\mathfrak{P}}$-group acting injectively on Y, then the following statements hold.*

1. *The group X is $\mathcal{F}_{\mathfrak{P}}$-quasi-regular with respect to Y.*
2. *If Y lies in the center of X, then X is $\mathcal{F}_{\mathfrak{P}}$-regular with respect to Y.*

Proposition 4.4. *Suppose that $\mathfrak{P}$ is a non-empty set of primes, X is a group, and Y is a subgroup of X. Suppose also that at least one of the following conditions holds:*

(α) *$X \in \mathcal{BN}_{\mathfrak{P}}$;*

(β) *X is residually a $\mathfrak{P}'$-torsion-free $\mathcal{BN}_{\mathfrak{P}}$-group and has a homomorphism onto such a group acting injectively on Y.*

Then the set $\mathfrak{P}'$-$\mathfrak{Rt}(X, Y)$ is a subgroup, which is $\mathcal{FN}_{\mathfrak{P}}$-separable in X. In particular, if X is $\mathfrak{P}'$-torsion-free, then it is residually an $\mathcal{FN}_{\mathfrak{P}}$-group.

Proposition 4.5. *If $\mathfrak{P}$ is a finite set of primes and X is a periodic $\mathfrak{P}'$-torsion-free $\mathcal{BN}_{\mathfrak{P}}$-group, then X is finite.*

Proof. If X is abelian, then it belongs to the class $\mathcal{BA}_{\mathfrak{P}}$ by Proposition 4.2. In accordance with the definition of this class, a primary component of X is finite if it corresponds to a number from $\mathfrak{P}$. It follows that X is also finite because it is $\mathfrak{P}'$-torsion-free and $\mathfrak{P}$ is finite. In the general case, X has a finite central series with $\mathcal{BA}_{\mathfrak{P}}$-factors, which are finite as proven above. Therefore, X is finite again.

5. Generalized free products. Theorem 6

Recall that *the generalized free product of groups A and B with subgroups $H \leqslant A$ and $K \leqslant B$ amalgamated under an isomorphism $\varphi\colon H \to K$* is the group G defined as follows:

– the generators of G are the generators of A and B;

– the defining relations of G are the relations of A and B together with all possible relations of the form $h = h\varphi$, where h and $h\varphi$ are some words in the generators of A and B that define an element of H and its image under φ.

It is well known that the free factors A and B can be embedded in G via the identical mappings of their generators. This fact allows us to consider A and B subgroups of G. Under this assumption, the subgroups H and K turn out to be equal. Thus, we can use the notation $G = \langle A {*} B;\ H\rangle$ and say that G is *the generalized free product of the groups A and B with the subgroup H amalgamated.*

Let $G = \langle A * B;\ H\rangle$. Recall that the representation of an element $g \in G$ as a product $g_1 g_2 \dots g_n$ is said to be a *reduced form* of this element if $n \geqslant 1$, $g_1, g_2, \dots, g_n \in A \cup B$, and, for $n > 1$, no two adjacent factors of the product (called *syllables* of the reduced form) belong simultaneously to A or B. By the normal form theorem for generalized free products (see, for example, [35, Chapter IV, Theorem 2.6]), if g has a reduced form of length greater than 1, then $g \neq 1$. It follows that all reduced forms of g have the same length, which is denoted below by $\ell(g)$.

If R and S are normal subgroups of A and B, $R \cap H = S \cap H$, and $\varphi_{R,S}\colon HR/R \to HS/S$ is the mapping taking the coset hR to the coset hS, then this mapping is well defined and is an isomorphism of the subgroups HR/R and HS/S. Therefore, we can consider the generalized free product $G_{R,S}$ of the groups A/R and B/S with the subgroups HR/R and HS/S amalgamated under the isomorphism $\varphi_{R,S}$. It is easy to see that the natural homomorphisms $A \to A/R$ and $B \to B/S$ can be extended to a surjective homomorphism $\rho_{R,S}\colon G \to G_{R,S}$ and the kernel of the latter is the normal closure in G of the set $R \cup S$. We note also that if X is a normal subgroup of G, $R = X \cap A$, and $S = X \cap B$, then R and S are normal subgroups of the groups A and B, respectively, $R \cap H = S \cap H$, and therefore the group $G_{R,S}$ and the homomorphism $\rho_{R,S}$ are defined.

The next proposition is a special case of Theorem 5 from [29].

Proposition 5.1. *Let $G = \langle A * B;\ H\rangle$. If a normal subgroup N of G intersects H trivially, then it can be decomposed into the (ordinary) free product of a free subgroup and subgroups, each of which is conjugate to $N \cap A$ or $N \cap B$.*

Proposition 5.2. *Suppose that $G = \langle A * B;\ H\rangle$ and $\mathcal{C}$ is a class of groups. If*

$$(\alpha)\qquad \bigcap_{X \in \mathcal{C}^*(G)} X \cap A = 1 = \bigcap_{X \in \mathcal{C}^*(G)} X \cap B,$$

$$(\beta)\qquad \bigcap_{X \in \mathcal{C}^*(G)} H(X \cap A) = H = \bigcap_{X \in \mathcal{C}^*(G)} H(X \cap B),$$

$$(\gamma)\qquad \forall X, Y \in \mathcal{C}^*(G)\ \exists Z \in \mathcal{C}^*(G)\ Z \leqslant X \cap Y,$$

then the following statements hold.

1. *For each element $g \in G \setminus \{1\}$, which is conjugate to some element of $A \cup B$, there exists a homomorphism of G onto a group from $\mathcal{C}$ taking g to a non-trivial element.*

2. *For each element $g \in G \setminus \{1\}$, which is conjugate to no element of $A \cup B$, there exists a homomorphism of G onto a group from $\Phi \cdot \mathcal{C}$ taking g to a non-trivial element.*

In particular, G is residually a $\Phi \cdot \mathcal{C}$-groups (recall that Φ denotes the class of all free groups).

Proof. Let $g \in G \setminus \{1\}$. If g is conjugate to some element $a \in A$, then, by (α), there exists a subgroup $X \in \mathcal{C}^*(G)$ that does not contain a. It is clear that $g \notin X$, and therefore the natural homomorphism $G \to G/X$ is the desired one. A similar argument can be used if g is conjugate to an element of B. Thus, Statement 1 is proved.

If g is conjugate to no element of $A \cup B$ and $g_1 g_2 \ldots g_n$ is its reduced form, then $n > 1$ and therefore $g_1, g_2, \ldots, g_n \in A \setminus H \cup B \setminus H$. It follows from (β) that, for each $i \in \{1, \ldots, n\}$, there exists a subgroup $X_i \in \mathcal{C}^*(G)$ such that $g_i \notin H(X_i \cap A)$ if $g_i \in A \setminus H$, and $g_i \notin H(X_i \cap B)$ if $g_i \in B \setminus H$. By (γ), the intersection $X_1 \cap X_2 \cap \ldots \cap X_n$ contains a subgroup $X \in \mathcal{C}^*(G)$. Let $R = X \cap A$ and $S = X \cap B$. Then $g_i \notin HR$ if $g_i \in A \setminus H$, and $g_i \notin HS$ if $g_i \in B \setminus H$, where $1 \leqslant i \leqslant n$. It follows that $(g_1\rho_{R,S})(g_2\rho_{R,S}) \ldots (g_n\rho_{R,S})$ is a reduced form of the element $g\rho_{R,S}$ of length $n > 1$ and therefore $g\rho_{R,S} \neq 1$. Since $\rho_{R,S}$ continues the natural homomorphisms $A \to A/R$, $B \to B/S$ and the kernel of $\rho_{R,S}$ is contained in X as the normal closure of the set $R \cup S = (X \cap A) \cup (X \cap B)$, we have $G_{R,S}/X\rho_{R,S} \cong G/X \in \mathcal{C}$ and

$$X\rho_{R,S} \cap A/R = X\rho_{R,S} \cap A\rho_{R,S} = 1 = X\rho_{R,S} \cap B\rho_{R,S} = X\rho_{R,S} \cap B/S.$$

It follows from the last equalities and Proposition 5.1 that $X\rho_{R,S}$ is a free group and $\rho_{R,S}$ is the desired homomorphism.

Let us recall [46] that a class of groups $\mathcal{C}$ is said to be a *root class* if it contains non-trivial groups, is closed under taking subgroups, and satisfies any of the following three equivalent conditions:

(α) for any group X and for any subnormal series $1 \leqslant Z \leqslant Y \leqslant X$, if $X/Y, Y/Z \in \mathcal{C}$, then there exists a subgroup $T \in \mathcal{C}^*(X)$ such that $T \leqslant Z$ (the *Gruenberg condition*);

(β) the class $\mathcal{C}$ is closed under taking unrestricted wreath products;

(γ) the class $\mathcal{C}$ is closed under taking extensions and, for any two groups $X, Y \in \mathcal{C}$, contains the unrestricted direct product $\prod_{y \in Y} X_y$, where X_y is an isomorphic copy of X for each $y \in Y$.

It is easy to see that $\mathcal{F}_{\mathfrak{P}}$ is a root class for any non-empty set of prime numbers $\mathfrak{P}$. The classes $\mathcal{FN}_{\mathfrak{P}}$ (if $\mathfrak{P}$ contains at least two prime numbers) and $\mathcal{BN}_{\mathfrak{P}}$ are not root classes since they are not closed under taking extensions. The next proposition follows from Theorems 1, 3 and Proposition 2 given in [47].

Proposition 5.3. *Suppose that $G = \langle A * B;\ H \rangle$ and $\mathcal{C}$ is a root class of groups. If A and B are residually $\mathcal{C}$-groups, H is $\mathcal{C}$-separable in these groups, and G is $\mathcal{C}$-quasi-regular with respect to A and B, then G is residually a $\mathcal{C}$-group.*

Proposition 5.4. [54, Theorem 3] *Suppose that $G = \langle A * B;\ H \rangle$ and $\mathcal{C}$ is a root class of groups closed under taking quotient groups. Suppose also that H is finite and normal in A and B. Then G is residually a $\mathcal{C}$-group if and only if $\operatorname{Aut}_G(H) \in \mathcal{C}$.*

The next proposition is obtained by combining Proposition 3 and 6 from [34].

Proposition 5.5. *Suppose that $G = \langle A * B;\ H \rangle$ and p is a prime. Suppose also that*

$$R = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A \quad \textit{and} \quad S = S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B$$

are sequences of subgroups of A and B such that

1) $R_i \in \mathcal{F}_p^*(A)$, $S_i \in \mathcal{F}_p^*(B)$, $0 \leqslant i \leqslant n$;

2) $R_i \cap H = S_i \cap H$, $0 \leqslant i \leqslant n$;

3) $|(R_{i+1} \cap H)/(R_i \cap H)| \in \{1, p\}$, $0 \leqslant i \leqslant n-1$.

Then $G_{R,S}$ is residually an $\mathcal{F}_p$-group.

Proposition 5.6. *Suppose that $G = \langle A * B;\ H \rangle$ and U is a subgroup of H, which is normal in G. Suppose also that λ and μ are homomorphisms of the groups A and B, respectively, which map them onto nilpotent groups and act injectively on H. Then, for each prime number p, the subgroup $R = p'\text{-}\mathfrak{Is}(A, U \cdot \ker \lambda)$ is normal in A, the subgroup $S = p'\text{-}\mathfrak{Is}(B, U \cdot \ker \mu)$ is normal in B, and $R \cap H = p'\text{-}\mathfrak{Rt}(H, U) = S \cap H$.*

Proof. Since $A\lambda$ and $B\mu$ are nilpotent, the sets $p'\text{-}\mathfrak{Rt}(A\lambda, U\lambda)$ and $p'\text{-}\mathfrak{Rt}(B\mu, U\mu)$ are subgroups by Proposition 4.1. It is easy to see that the sets $p'\text{-}\mathfrak{Rt}(A, U \cdot \ker \lambda)$

and $p'\text{-}\mathfrak{Rt}(B,\, U\cdot\ker\mu)$ are the full pre-images of these subgroups under λ and μ, respectively. It follows that

$$R = p'\text{-}\mathfrak{Rt}(A,\, U\cdot\ker\lambda), \qquad R\cap H = p'\text{-}\mathfrak{Rt}(H,\, (U\cdot\ker\lambda)\cap H),$$
$$S = p'\text{-}\mathfrak{Rt}(B,\, U\cdot\ker\mu), \qquad S\cap H = p'\text{-}\mathfrak{Rt}(H,\, (U\cdot\ker\mu)\cap H),$$

and the subgroups R and S are normal in A and B, respectively, because the subgroups $U\cdot\ker\lambda$ and $U\cdot\ker\mu$ have the same property. It remains to note that since $\ker\lambda\cap H = 1 = \ker\mu\cap H$, the equalities

$$(U\cdot\ker\lambda)\cap H = U = (U\cdot\ker\mu)\cap H$$

hold.

Everywhere below, if the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$, then the phrase "λ and μ are homomorphisms which exist due to this condition" means that λ is a homomorphism of A, μ is a homomorphism of B, $A\lambda, B\mu \in \mathcal{BN}_{\mathfrak{P}}$, and $\ker\lambda\cap H = \ker\mu\cap H = 1$.

Suppose that p is a prime number and the symbols $U(p)$, $V(p)$, and $W(p)$ denote the subgroups H^pH', $p'\text{-}\mathfrak{Is}(A,\, U(p)\cdot\ker\lambda)$, and $p'\text{-}\mathfrak{Is}(B,\, U(p)\cdot\ker\mu)$, respectively. If H is normal in A and B, then $U(p)$ is normal in G. By Proposition 5.6, it follows that $V(p)$ and $W(p)$ are normal in A and B, respectively, and since $U(p)$ is obviously p'-isolated in H, the equalities $V(p)\cap H = U(p) = W(p)\cap H$ hold. Thus, we can consider the group $G_{V(p),W(p)}$, the homomorphism $\rho_{V(p),W(p)}$, and also the group $\mathrm{Aut}_{G_{V(p),W(p)}}(H\rho_{V(p),W(p)})$, which is correctly defined because $H\rho_{V(p),W(p)}$ is normal in $G_{V(p),W(p)}$.

Theorem 6. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$, λ and μ are homomorphisms which exist due to this condition, and H is normal in A and B. Suppose also that, for each $p\in\mathfrak{P}$, the symbols $U(p)$, $V(p)$, and $W(p)$ denote the subgroups H^pH', $p'\text{-}\mathfrak{Is}(A,\, U(p)\cdot\ker\lambda)$, and $p'\text{-}\mathfrak{Is}(B,\, U(p)\cdot\ker\mu)$, respectively. If $\mathrm{Aut}_{G_{V(p),W(p)}}(H\rho_{V(p),W(p)})$ is a p-group, then Statements 1—3 of Theorem 2 hold.*

6. Necessary conditions for the residual nilpotence. Proof of Theorem 5

Proposition 6.1. *Suppose that X is a group, q and r are prime numbers, and $x_1, x_2\in X$. Suppose also that, for any prime p and for any homomorphism σ of X onto an $\mathcal{F}_p$-group, if $p\neq q$, then $x_1\sigma = 1$, and if $p\neq r$, then $x_2\sigma = 1$. If $q\neq r$ and $[x_1, x_2]\neq 1$, then X is not residually a finite nilpotent group.*

Proof. Assume that $q\neq r$, $[x_1, x_2]\neq 1$, and X is residually a finite nilpotent group. By Proposition 3.7, X is residually a $\mathcal{C}$-group, where $\mathcal{C}$ is the union of the classes $\mathcal{F}_p$

over all primes p. Hence, there exist a prime number p and a homomorphism σ of X onto an $\mathcal{F}_p$-group taking $[x_1, x_2]$ to a non-trivial element. Since $q \neq r$, at least one of the inequalities $p \neq q$ and $p \neq r$ holds. Therefore, $x_1\sigma = 1$ or $x_2\sigma = 1$, and, in both cases, $[x_1, x_2]\sigma = 1$, contrary to the choice of σ.

Proposition 6.2. *The group*

$$G = \langle a, b, c;\ a^9 = [a^3, b] = [a^3, c] = c^{-1}bcb = 1\rangle$$

is not residually a finite nilpotent group.

Proof. Let p and σ be a prime number and a homomorphism of G onto an $\mathcal{F}_p$-group. It is clear that if $p \neq 3$, then $a\sigma = 1$. If $p \neq 2$, then the orders r and s of $b\sigma$ and $c\sigma$ are odd. Hence, $b\sigma = (c\sigma)^{-s}(b\sigma)(c\sigma)^s = (b\sigma)^{(-1)^s} = (b\sigma)^{-1}$, $(b\sigma)^2 = 1 = (b\sigma)^r$, and $b\sigma = 1$. If G is considered the generalized free product of the groups $A = \mathrm{sgp}\{a\}$ and $B = \mathrm{sgp}\{a^3, b, c\}$ with the subgroup $H = \mathrm{sgp}\{a^3\}$ amalgamated, then $[a, b]$ is of length 4 and therefore is non-trivial. By Proposition 6.1, it follows that G is not residually a finite nilpotent group.

The next proposition is given without proof in [7] and can be easily verified by induction on k.

Proposition 6.3. *If*

$$\begin{aligned} \upsilon_1(x, y) &= [x, y] = x^{-1}y^{-1}xy, \\ \upsilon_{k+1}(x, y) &= [x, \upsilon_k(x, y)] = x^{-1}\upsilon_k(x, y)^{-1}x\upsilon_k(x, y), \quad k \geqslant 1, \end{aligned} \tag{1}$$

then, for each $k \geqslant 1$, the word $\upsilon_k(x, y)$ in the alphabet $\{x^{\pm 1}, y^{\pm 1}\}$ is of length 2^{k+1}, starts with $x^{-1}y^{-1}$, ends with xy, and does not contain a subword of the form $x^{\varepsilon}x^{\delta}$ or $y^{\varepsilon}y^{\delta}$, where $\varepsilon, \delta = \pm 1$.

Proposition 6.4. [44, Lemma 2] *If a group X satisfies a non-trivial identity, then it satisfies a non-trivial identity of the form*

$$\omega(y, x_1, x_2) = \omega_0(x_1, x_2)y^{\varepsilon_1}\omega_1(x_1, x_2)\ldots y^{\varepsilon_n}\omega_n(x_1, x_2), \tag{2}$$

where $n \geqslant 1$, $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$, and $\omega_0(x_1, x_2), \ldots, \omega_n(x_1, x_2) \in \{x_1^{\pm 1}, x_2^{\pm 1}, (x_1x_2^{-1})^{\pm 1}\}$.

Proof of Theorem 5. By the condition of the theorem, there exists a subgroup M of A such that H is properly contained in M and M satisfies a non-trivial identity or lies in the normalizer of H. Let N denote the subgroup of B defined in the same way. We prove both statements of the theorem by contradiction. Let us begin with Statement 2 and assume that H is not $\mathcal{C}$-separable in A, i.e., there exists an element $u \in A \setminus H$ such that $u\theta \in H\theta$ for any homomorphism θ of A onto a $\mathcal{C}$-group.

If H is normal in N, we fix an element $b \in N \setminus H$ and put $g_1(u, b) = \upsilon_c(u, b^{-1}ub)$, where c is the nilpotency class of H and $\upsilon_c(x, y)$ is an element of the sequence (1). Otherwise, $[N : H] \geqslant 3$ and N satisfies a non-trivial identity, which can be considered having the form (2). In this case, we choose elements $b_1, b_2 \in N \setminus H$ that belong to different right cosets of H in N and put $g_2(u, b_1, b_2) = \omega(u, b_1, b_2)$.

It follows from Propositions 6.3, 6.4 and the relation $b_1 b_2^{-1} \in N \setminus H$ that, in the generalized free product G, the elements $g_1(u, b)$ and $g_2(u, b_1, b_2)$ have reduced forms of length greater than 1. Therefore, they are non-trivial. But if σ is a homomorphism of G onto a $\mathcal{C}$-group, then $A\sigma \in \mathcal{C}$ because $\mathcal{C}$ is closed under taking subgroups, and $u\sigma \in H\sigma$ due to the choice of u. Hence,

a) if H is normal in N, then $g_1(u, b)\sigma = \upsilon_c(u\sigma, (b\sigma)^{-1}u\sigma b\sigma) = 1$ since $u\sigma \in H\sigma$, $(b\sigma)^{-1}u\sigma b\sigma \in H\sigma$, the element $\upsilon_c(u\sigma, (b\sigma)^{-1}u\sigma b\sigma)$ belongs to the $(c + 1)$-th member of the lower central series of $H\sigma$, and the nilpotency class of this group does not exceed c;

b) otherwise, $g_2(u, b_1, b_2)\sigma = \omega(u\sigma, b_1\sigma, b_2\sigma) = 1$ since $u\sigma \in N\sigma$, $b_1\sigma \in N\sigma$, $b_2\sigma \in N\sigma$ and $N\sigma$ satisfies $\omega(y, x_1, x_2)$.

Thus, we get a contradiction with the fact that G is residually a $\mathcal{C}$-group. The $\mathcal{C}$-separability of H in B can be proved in the same way.

Now let us turn to the proof of Statement 1 and assume that, for any prime number p, H is not p'-isolated in A and B. It follows that there exist elements $u, v \in A \setminus H \cup B \setminus H$ and prime numbers q, r satisfying the relations $q \neq r$, $u^q \in H$, and $v^r \in H$. We need to find elements $g_1, g_2 \in G$ such that the commutator $[g_1, g_2]$ has a reduced form of length greater than 1 (and therefore is non-trivial), but, for any homomorphism σ of G onto an $\mathcal{F}_p$-group, $g_1\sigma = 1$ if $p \neq q$, and $g_2\sigma = 1$ if $p \neq r$. By Proposition 6.1, the existence of such elements means that G is not residually an $\mathcal{F}\mathcal{N}_{\mathfrak{P}}$-group, contrary to the condition of the theorem.

Since $q \neq r$, at least one of these numbers does not equal 2. Without lost of generality we can assume that $r \geqslant 3$ and $u \in A \setminus H$. As above, if H is normal in M (in N), we fix an element $a \in M \setminus H$ ($b \in N \setminus H$). Otherwise, we choose elements $a_1, a_2 \in M \setminus H$ ($b_1, b_2 \in N \setminus H$) lying in different right cosets of H in M (in N) and denote by $\omega_M(y, x_1, x_2)$ ($\omega_N(y, x_1, x_2)$) the non-trivial identity of the form (2), which M (respectively N) satisfies. Below, we define the elements g_1 and g_2 in each of the following cases independently.

Case 1. $v \in A \setminus H$ and H is normal in N.

Since all the elements v, v^2, ..., v^{r-1} lie in different right cosets of H in A and $r \geqslant 3$, there exists $k \in \{1, 2, \dots, r - 1\}$ such that $v^k u^{-1} \in A \setminus H$. We put $g_1 = \upsilon_c(u, b^{-1}ub)$ and $g_2 = \upsilon_c(v^k, b^{-1}v^k b)$, where c is the nilpotency class of H and $\upsilon_c(x, y)$ is an element of the sequence (1).

Case 2. $v \in A \setminus H$ and H is not normal in N.

Let $g_1 = \omega_N(u, b_1, b_2)$ and $g_2 = v^{-1}\omega_N(v, b_1, b_2)v$.

Case 3. $v \in B \setminus H$ and H is normal in M and N.

As in Case 1, there exists $m \in \{1, 2, \dots, r - 1\}$ such that $v^m b^{-1} \in B \setminus H$. We put $g_1 = \upsilon_c(b^{-1}ub, u)$ and $g_2 = \upsilon_c(v^m, a^{-1}v^m a)$, where c and $\upsilon_c(x, y)$ are defined as above.

Case 4. $v \in B \setminus H$ and H is normal in M, but not in N.

Let $g_1 = u^{-1}\omega_N(u, b_1, b_2)u$ and $g_2 = \upsilon_c(v, a^{-1}va)$.

Case 5. $v \in B \setminus H$ and H is normal in N, but not in M.

Let $g_1 = \upsilon_c(u, b^{-1}ub)$ and $g_2 = v^{-1}\omega_M(v, a_1, a_2)v$.

Case 6. $v \in B \setminus H$ and H is not normal both in M and in N.

Let $g_1 = \omega_N(u, b_1, b_2)$ and $g_2 = \omega_M(v, a_1, a_2)$.

It follows from the relations

$$v^k u^{-1} \in A \setminus H, \quad v^m b^{-1} \in B \setminus H, \quad a_1 a_2^{-1} \in M \setminus H, \quad b_1 b_2^{-1} \in N \setminus H$$

and Propositions 6.3, 6.4 that

– in each of Cases 1—6, g_1 and g_2 have reduced forms of length greater than 1;

– in Cases 2 and 6, there are no cancellations at the boundaries of the multiplied elements g_1^{-1}, g_2^{-1}, g_1, g_2, and therefore $\ell(g_1^{-1}g_2^{-1}g_1g_2) = 2(\ell(g_1) + \ell(g_2)) > 1$;

– in Cases 1 and 3, the boundary syllables of the words g_2^{-1} and g_1 are combined into one, whence

$$\ell(g_1^{-1}g_2^{-1}g_1g_2) = 2(\ell(g_1) + \ell(g_2)) - 1 > 1;$$

– in Case 4, $\ell(g_2^{-1}g_1g_2) = \ell(g_1) + 2\ell(g_2)$ and $\ell(g_1^{-1}g_2^{-1}g_1g_2) \geqslant 2\ell(g_2) > 1$;

– in Case 5, $\ell(g_1^{-1}g_2^{-1}g_1) = 2\ell(g_1) + \ell(g_2)$ and $\ell(g_1^{-1}g_2^{-1}g_1g_2) \geqslant 2\ell(g_1) > 1$.

It is easy to see that, given a prime p and a homomorphism σ of G onto an $\mathcal{F}_p$-group, we have $u\sigma \in H\sigma$ if $p \neq q$, and $v\sigma \in H\sigma$ if $p \neq r$. It follows from this fact and from the definitions of c, υ_c, ω_M, and ω_N that, in each of the considered cases, $g_1\sigma = 1$ if $p \neq q$, and $g_2\sigma = 1$ if $p \neq r$, as required.

Thus, H is p'-isolated in A and B for some prime number p. By Statement 2 of this theorem, H is also $\mathfrak{P}'$-isolated in these groups. If $p \notin \mathfrak{P}$, it follows that H is isolated and, hence, q'-isolated in A and B for any $q \in \mathfrak{P}$. Thus, Statement 1 is completely proved.

Proposition 6.5. *If $G = \langle A * B;\ H\rangle$, $A \neq H \neq B$, and H is periodic, then Statement* 1 *of Theorem* 5 *holds.*

Proof. Let us assume that G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, but there exist elements $u, v \in A \setminus H \cup B \setminus H$ and prime numbers q, r such that $q \neq r$ and $u^q, v^r \in H$. The orders of u and v can be decomposed into the products $s \cdot s'$ and $t \cdot t'$, where s is a q-number, s' is a q'-number, t is an r-number, and t' is an r'-number. Since q and s' are co-prime, the relation $u^{s'} \in H$ is equivalent to the inclusion $u \in H$, which contradicts the choice of u. Hence, $u^{s'} \notin H$ and similarly $v^{t'} \notin H$. Using the condition $A \neq H \neq B$, we can choose some elements $a \in A \setminus H$, $b \in B \setminus H$ and put $x_1 = u^{s'}$,

$$x_2 = \begin{cases} v^{t'} & \text{if } u \in A \setminus H \text{ and } v \in B \setminus H \text{ or } v \in A \setminus H \text{ and } u \in B \setminus H; \\ b^{-1}v^{t'}b & \text{if } u, v \in A \setminus H; \\ a^{-1}v^{t'}a & \text{if } u, v \in B \setminus H. \end{cases}$$

It is easy to see that $\ell([x_1, x_2]) = 4$ if $x_2 = v^{t'}$, and $\ell([x_1, x_2]) = 8$ otherwise. Thus, $[x_1, x_2] \neq 1$. At the same time, it follows from the equalities $x_1^s = 1 = x_2^t$ and the definitions of s and t that if p and σ are a prime number and a homomorphism of G onto an $\mathcal{F}_p$-group, then $x_1\sigma = 1$ whenever $p \neq q$, and $x_2\sigma = 1$ whenever $p \neq r$. Therefore, by Proposition 6.1, G is not residually a finite nilpotent group, contrary to the assumption.

Thus, we prove that if G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then H is p'-isolated in A and B for some prime number p. In addition, if G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, then it is $\mathfrak{P}'$-torsion-free and therefore H is $\mathfrak{P}'$-isolated in the free factors. Hence, the inclusion $p \in \mathfrak{P}$ can be proved in the same way as in the proof of Theorem 5.

7. Conditions for the quasi-regularity

Proposition 7.1. *Suppose that $G = \langle A * B;\ H \rangle$ and $\mathcal{C}$ is a root class of groups. If G is $\mathcal{C}$-quasi-regular with respect to H, then it is $\mathcal{C}$-quasi-regular with respect to A and B.*

Proof. Let M be a subgroup from $\mathcal{C}^*(A)$. Since $\mathcal{C}$ is closed under taking subgroups, it follows from Proposition 3.1 that $M \cap H \in \mathcal{C}^*(H)$. Due to the $\mathcal{C}$-quasi-regularity of G with respect to H, there exists a subgroup $U \in \mathcal{C}^*(G)$ satisfying the relation $U \cap H \leqslant M \cap H$. Let us put $Q = U \cap A$, $R = U \cap A \cap M$, and $S = U \cap B$. Then $Q, R \in \mathcal{C}^*(A)$ and $S \in \mathcal{C}^*(B)$ by Proposition 3.1.

The inclusion $U \cap H \leqslant M \cap H$ implies that $R \cap H = U \cap H = S \cap H$. Hence, the group $G_{R,S}$ and the homomorphism $\rho_{R,S}$ are defined. Recall that the latter continues the natural homomorphisms $A \to A/R$, $B \to B/S$ and the subgroup $\ker \rho_{R,S}$ coincides with the normal closure of $R \cup S$ in G. It follows that

$$\ker \rho_{R,S} \leqslant U, \quad G_{R,S}/U\rho_{R,S} \cong G/U \in \mathcal{C}, \quad Q\rho_{R,S} \cong QR/R,$$
$$U\rho_{R,S} \cap A\rho_{R,S} = Q\rho_{R,S}, \quad \text{and} \quad U\rho_{R,S} \cap B\rho_{R,S} = 1.$$

By Proposition 5.1, the relations $U\rho_{R,S} \cap H\rho_{R,S} \leqslant U\rho_{R,S} \cap B\rho_{R,S} = 1$ mean that $U\rho_{R,S}$ can be decomposed into the (ordinary) free product of a free subgroup F and subgroups Q_i, $i \in \mathcal{I}$, each of which is conjugate to $Q\rho_{R,S}$. Let us denote by σ the homomorphism of $U\rho_{R,S}$ onto $Q\rho_{R,S}$ which continues the isomorphisms $Q_i \to Q\rho_{R,S}$, $i \in \mathcal{I}$, and takes the generators of F to 1.

Since $Q\rho_{R,S} \cong QR/R \leqslant A/R \in \mathcal{C}$ and $\mathcal{C}$ is closed under taking subgroups, we have $U\rho_{R,S}\sigma \in \mathcal{C}$. Thus, the class $\mathcal{C}$ contains the factors $G_{R,S}/U\rho_{R,S}$ and $U\rho_{R,S}/\ker\sigma$ of the subnormal series $1 \leqslant \ker\sigma \leqslant U\rho_{R,S} \leqslant G_{R,S}$. By the Gruenberg condition (see the definition of root class in Section 5), this implies the existence of a subgroup

$N_{R,S} \in \mathcal{C}^*(G_{R,S})$ such that $N_{R,S} \leqslant \ker\sigma$. Since σ acts injectively on the subgroups Q_i, $i \in \mathcal{I}$, which are conjugate to $Q\rho_{R,S}$, we have $N_{R,S} \cap Q\rho_{R,S} = 1$. It follows from this equality and the relations $U\rho_{R,S} \cap A\rho_{R,S} = Q\rho_{R,S}$ and $N_{R,S} \leqslant U\rho_{R,S}$ that $N_{R,S} \cap A\rho_{R,S} = 1$.

Let N denote the pre-image of the subgroup $N_{R,S}$ under the homomorphism $\rho_{R,S}$. Then $N \in \mathcal{C}^*(G)$, and since $N_{R,S} \cap A\rho_{R,S} = 1$, we have $N \cap A = R \leqslant M$. Therefore, G is $\mathcal{C}$-quasi-regular with respect to A. The $\mathcal{C}$-quasi-regularity of G with respect to B can be proved in the same way.

Proposition 7.2. *If $G = \langle A * B;\ H\rangle$, p is a prime number, A is $\mathcal{F}_p$-quasi-regular with respect to H, and B is $\mathcal{F}_p$-regular with respect to H, then G is $\mathcal{F}_p$-quasi-regular with respect to H.*

Proof. Let M be a subgroup from $\mathcal{F}_p^*(H)$. Since A is $\mathcal{F}_p$-quasi-regular with respect to H, there exists a subgroup $R \in \mathcal{F}_p^*(A)$ satisfying the relation $R \cap H \leqslant M$. It is well known that every finite p-group has a normal series with factors of order p. This implies the existence of a sequence of subgroups $R = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A$ such that $R_i \in \mathcal{F}_p^*(A)$, $0 \leqslant i \leqslant n$, and $|R_{i+1}/R_i| = p$, $0 \leqslant i \leqslant n-1$.

By Proposition 3.1, $R_i \cap H \in \mathcal{F}_p^*(H)$, $0 \leqslant i \leqslant n$. Since B is $\mathcal{F}_p$-regular with respect to H, for each $i \in \{0, \ldots, n\}$, there exists a subgroup $T_i \in \mathcal{F}_p^*(B)$ satisfying the equality $T_i \cap H = R_i \cap H$. Without lost of generality, we can assume that $T_n = B$. If $S_i = \bigcap_{j=i}^n T_j$, $0 \leqslant i \leqslant n$, then $S_i \in \mathcal{F}_p^*(B)$ by Proposition 3.1, $S_n = B$, $S_i \leqslant S_{i+1}$, $0 \leqslant i \leqslant n-1$, and

$$S_i \cap H = \bigcap_{j=i}^{n} T_j \cap H = \bigcap_{j=i}^{n} R_j \cap H = R_i \cap H.$$

In addition,

$$R_{i+1} \cap H/R_i \cap H \cong (R_{i+1} \cap H)R_i/R_i \leqslant R_{i+1}/R_i, \quad |R_{i+1}/R_i| = p, \quad 0 \leqslant i \leqslant n-1.$$

Therefore, all the conditions of Proposition 5.5 hold for the sequences of subgroups

$$R = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A \quad \text{and} \quad S = S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B,$$

and thus $G_{R,S}$ is residually an $\mathcal{F}_p$-group.

Since $\rho_{R,S}$ continues the natural homomorphism $A \to A/R$ and $A/R \in \mathcal{F}_p$, the subgroup $H\rho_{R,S}$ is finite. By Proposition 3.1, this implies the existence of a subgroup $N_{R,S} \in \mathcal{F}_p^*(G_{R,S})$ satisfying the equality $N_{R,S} \cap H\rho_{R,S} = 1$. If N denotes the pre-image of $N_{R,S}$ under $\rho_{R,S}$, then $N \in \mathcal{F}_p^*(G)$ and $N \cap H = R \cap H \leqslant M$. Thus, G is $\mathcal{F}_p$-quasi-regular with respect to H.

Proposition 7.3. *If p is a prime number, X is residually an $\mathcal{F}_p$-group, and Y is a locally cyclic subgroup of X, then X is $\mathcal{F}_p$-regular with respect to Y.*

Proof. First of all, let us note that if $N \in \mathcal{F}_p^*(Y)$ and $k = [Y : N]$, then $N = Y^k$. Indeed, it is obvious that $Y^k \leqslant N$. At the same time, since the exponent of the locally cyclic group Y/Y^k divides k, this group is cyclic and has an order at most k. Hence, $Y^k = N$.

Suppose now that M is a subgroup from $\mathcal{F}_p^*(Y)$, yM is a generator of the finite cyclic subgroup Y/M, q is the order of the latter, and $S = \{y, y^2, \ldots, y^{q-1}\}$. Then $1 \notin S$ and since X is residually an $\mathcal{F}_p$-group, Proposition 3.1 implies the existence of a subgroup $Z \in \mathcal{F}_p^*(X)$ satisfying the relation $S \cap Z = \varnothing$. It is clear that any two non-equal elements of S lie in different cosets of Z. Therefore, $q \leqslant r$, where $r = |YZ/Z| = [Y : Z \cap Y]$. Since q and r are p-numbers, $M = Y^q$, and $Z \cap Y = Y^r$, it follows that q divides r, $Z \cap Y \leqslant M$,

$$(YZ/Z)/(MZ/Z) \cong Y/M(Y \cap Z) = Y/M,$$

and $[YZ/Z : MZ/Z] = q$. If

$$1 = Z_0/Z \leqslant Z_1/Z \leqslant \ldots \leqslant Z_n/Z = X/Z$$

is a normal series of the $\mathcal{F}_p$-group X/Z with factors of order p, then the factors of the series

$$1 = YZ/Z \cap Z_0/Z \leqslant YZ/Z \cap Z_1/Z \leqslant \ldots \leqslant YZ/Z \cap Z_n/Z = YZ/Z$$

are of order 1 or p. Therefore,

$$\big\{[YZ/Z : YZ/Z \cap Z_i/Z] \mid 0 \leqslant i \leqslant n\big\} = \big\{1, p, p^2, \ldots, r\big\}.$$

Since the finite cyclic group YZ/Z contains only one subgroup of index q, we have $Z_i/Z \cap YZ/Z = MZ/Z$ for some $i \in \{0, 1, \ldots, n\}$. It easily follows from the last equality and the inclusion $Z \cap Y \leqslant M$ that $Z_i \cap Y = M$. Since $Z_i/Z \in \mathcal{F}_p^*(X/Z)$, the relation $Z_i \in \mathcal{F}_p^*(X)$ holds. Thus, X is $\mathcal{F}_p$-regular with respect to Y.

Proposition 7.4. *Suppose that X is a group, Y is a subgroup of X, p is a prime number, and σ is a homomorphism of X such that $\ker \sigma \cap Y \leqslant p'\text{-}\mathfrak{Is}(Y, 1)$. If $X\sigma$ is $\mathcal{F}_p$-quasi-regular with respect to $Y\sigma$, then X is $\mathcal{F}_p$-quasi-regular with respect to Y.*

Proof. Let M be a subgroup from $\mathcal{F}_p^*(Y)$. Then $M\sigma \in \mathcal{F}_p^*(Y\sigma)$, and because $X\sigma$ is $\mathcal{F}_p$-quasi-regular with respect to $Y\sigma$, there exists a subgroup $N_\sigma \in \mathcal{F}_p^*(X\sigma)$ satisfying the relation $N_\sigma \cap Y\sigma \leqslant M\sigma$. It is clear that the pre-image N of N_σ under σ belongs to $\mathcal{F}_p^*(X)$. Since M is, obviously, p'-isolated in Y, the inclusion $p'\text{-}\mathfrak{Is}(Y, 1) \leqslant M$ holds. Now it follows from the relations $N_\sigma \cap Y\sigma \leqslant M\sigma$, $M \leqslant Y$, and $\ker \sigma \cap Y \leqslant p'\text{-}\mathfrak{Is}(Y, 1)$ that $N \cap Y \leqslant M(\ker \sigma \cap Y) = M$. Thus, X is $\mathcal{F}_p$-quasi-regular with respect to Y.

Proposition 7.5. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that λ and μ are homomorphisms which exist due to this condition, $p \in \mathfrak{P}$,* *and the symbols* $H(p)$, $U(p)$, $V(p)$, *and* $W(p)$ *denote the subgroups* p-$\mathfrak{Is}(H,1)$, H^pH', p'-$\mathfrak{Is}(A, U(p)\cdot\ker\lambda)$, *and* p'-$\mathfrak{Is}(B, U(p)\cdot\ker\mu)$, *respectively. Then* G *is* $\mathcal{F}_p$*-quasi-regular with respect to* H *if at least one of the following statements holds*:

(α) H *is locally cyclic*;

(β) H *lies in the center of* A *or* B;

(γ) H *is a retract of* A *or* B;

(δ) H *is periodic, and there exist sequences of subgroups*

$$1 = Q_0 \leqslant Q_1 \leqslant \ldots \leqslant Q_n = H(p), \quad R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A, \quad \textit{and}$$
$$S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B$$

such that

$$R_i \in \mathcal{F}_p^*(A), \quad S_i \in \mathcal{F}_p^*(B), \quad R_i \cap H(p) = Q_i = S_i \cap H(p), \quad 0 \leqslant i \leqslant n,$$
$$|Q_{i+1}/Q_i| = p, \quad 0 \leqslant i \leqslant n-1;$$

(ε) H *is normal in* A *and* B, *and the group* $\mathrm{Aut}_{G_{V(p),W(p)}}(H\rho_{V(p),W(p)})$, *which is defined due to Proposition* 5.6, *is a* p*-group.*

Proof. First of all, let us note that, by Condition ($*$), H can be embedded in a $\mathcal{BN}_{\mathfrak{P}}$-group and therefore is itself a $\mathcal{BN}_{\mathfrak{P}}$-group due to Proposition 4.2. We give the proof of the $\mathcal{F}_p$-quasi-regularity of G independently for each of the statements (α)—(ε).

(α) By Proposition 5.6, the subgroups $R = p'$-$\mathfrak{Is}(A, \ker\lambda)$ and $S = p'$-$\mathfrak{Is}(B, \ker\mu)$ are normal in A and B, respectively, and $R \cap H = p'$-$\mathfrak{Rt}(H,1) = S \cap H$. The groups $A\rho_{R,S} \cong A/R$ and $B\rho_{R,S} \cong B/S$ have no p'-torsion and are isomorphic to quotient groups of the $\mathcal{BN}_{\mathfrak{P}}$-groups $A\lambda$ and $B\mu$. Hence, they belong to $\mathcal{BN}_{\mathfrak{P}}$ due to Proposition 4.2. Since $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$, these groups are residually $\mathcal{F}_p$-groups by Proposition 4.4 and therefore they are $\mathcal{F}_p$-regular with respect to the locally cyclic subgroup $H\rho_{R,S}$ by Proposition 7.3. It now follows from Proposition 7.2 that $G_{R,S}$ is $\mathcal{F}_p$-quasi-regular with respect to $H\rho_{R,S}$. Since $\rho_{R,S}$ continues the natural homomorphism $A \to A/R$, we have

$$\ker\rho_{R,S} \cap H = R \cap H = p'\text{-}\mathfrak{Rt}(H,1) = p'\text{-}\mathfrak{Is}(H,1).$$

Hence, G is $\mathcal{F}_p$-quasi-regular with respect to H by Proposition 7.4.

(β) For definiteness, let H be central in B. If $R = \ker\lambda$ and $S = \ker\mu$, then $R \cap H = S \cap H = 1$ and therefore the group $G_{R,S}$ and the homomorphism $\rho_{R,S}$ are defined. Since $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$ and $\rho_{R,S}$ continues λ and μ, we have $A\rho_{R,S} \cong A\lambda \in \mathcal{BN}_p$, $B\rho_{R,S} \cong B\mu \in \mathcal{BN}_p$, and $\ker\rho_{R,S} \cap H = 1$. It follows from these relations and Propositions 4.3, 7.2, and 7.4 that $A\rho_{R,S}$ is $\mathcal{F}_p$-quasi-regular with respect to $H\rho_{R,S}$, $B\rho_{R,S}$ is $\mathcal{F}_p$-regular with respect to $H\rho_{R,S}$, $G_{R,S}$ is $\mathcal{F}_p$-quasi-regular with respect to $H\rho_{R,S}$, and G is $\mathcal{F}_p$-quasi-regular with respect to H.

(γ) For definiteness, let H be a retract of B, i.e., there exists a normal subgroup S of B satisfying the conditions $S \cap H = 1$ and $B = SH$. If, as above, $R = \ker\lambda$, then $R \cap H = 1 = S \cap H$ and $G_{R,S} = A\rho_{R,S}$. Since $A\rho_{R,S} \cong A\lambda \in \mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$, it follows from these relations and Propositions 4.3, 7.4 that $G_{R,S}$ is $\mathcal{F}_p$-quasi-regular with respect to $H\rho_{R,S}$ and G is $\mathcal{F}_p$-quasi-regular with respect to H.

(δ) Let M be a subgroup from $\mathcal{F}_p^*(H)$. By Proposition 4.1, the $\mathcal{BN}_{\mathfrak{P}}$-group H can be decomposed into the direct product of its subgroups $H(p)$ and $H(p') = p'\text{-}\mathfrak{Rt}(H, 1)$. It is clear that the subgroups M, R_i, and S_i, $0 \leqslant i \leqslant n$, are p'-isolated in H, A, and B, respectively. Therefore, $H(p') \leqslant M \cap R_i \cap S_i$, $0 \leqslant i \leqslant n$. Since $R_i \cap H(p) = Q_i = S_i \cap H(p)$, the equalities $R_i \cap H = H(p') \cdot Q_i = S_i \cap H$ hold. We also have

$$(H(p') \cdot Q_{i+1})/(H(p') \cdot Q_i) \cong Q_{i+1}/Q_i(H(p') \cap Q_{i+1}) = Q_{i+1}/Q_i,$$
$$|Q_{i+1}/Q_i| = p, \quad 0 \leqslant i \leqslant n-1.$$

Hence, if $R = R_0$ and $S = S_0$, then, by Proposition 5.5, $G_{R,S}$ is residually an $\mathcal{F}_p$-group. Since the group $A\rho_{R,S} \cong A/R$ is finite, Proposition 3.1 implies the existence of a subgroup $N_{R,S} \in \mathcal{F}_p^*(G_{R,S})$ satisfying the equality $N_{R,S} \cap A\rho_{R,S} = 1$. Let N denote the pre-image of $N_{R,S}$ under $\rho_{R,S}$. Then $N \in \mathcal{F}_p^*(G)$ and $N \cap A = \ker\rho_{R,S} \cap A = R$ because $\rho_{R,S}$ continues the natural homomorphism $A \to A/R$. It follows that $N \cap H = R \cap H = H(p') \leqslant M$ and G is $\mathcal{F}_p$-quasi-regular with respect to H.

(ε) Let again M be a subgroup from $\mathcal{F}_p^*(H)$. Suppose also that $r = [H : M]$ and $L = H^r$. As above, to complete the proof it is sufficient to indicate a subgroup $N \in \mathcal{F}_p^*(G)$ satisfying the condition $N \cap H \leqslant M$. If $r = 1$, then G is the required subgroup because $G \cap H = H = M$ and $G \in \mathcal{F}_p^*(G)$. Thus, we can assume that $r > 1$ and therefore $L \leqslant U(p)$. Let us put $R = p'\text{-}\mathfrak{Is}(A,\, L \cdot \ker\lambda)$ and $S = p'\text{-}\mathfrak{Is}(B,\, L \cdot \ker\mu)$. Since L is normal in G and p'-isolated in H, it follows from Proposition 5.6 that R is normal in A, S is normal in B, and $R \cap H = L = S \cap H$. The group H/L is periodic, has no p'-torsion, and satisfies the relations

$$H/L = H/H \cap R \cong HR/R \cong H\rho_{R,S} \in \mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$$

because $\rho_{R,S}$ continues the natural homomorphism $A \to A/R$, $H \in \mathcal{BN}_{\mathfrak{P}}$, and the class $\mathcal{BN}_{\mathfrak{P}}$ is closed under taking quotient groups due to Proposition 4.2. Hence, this group is finite by Proposition 4.5. Let us show that $\mathrm{Aut}_{G_{R,S}}(H\rho_{R,S}) \in \mathcal{F}_p$.

Since $U(p)$ is normal in G, the subgroup $U_{R,S} = U(p)\rho_{R,S}$ is normal in $G_{R,S}$. By Proposition 3.6, if $\alpha \in \mathrm{Aut}_{G_{R,S}}(H\rho_{R,S})$, $\overline{\alpha}$ is the automorphism of the group $H\rho_{R,S}/(H\rho_{R,S})^p(H\rho_{R,S})'$ induced by α, and the order of $\overline{\alpha}$ is a p-number, then the order of α is also a p-number. It is easy to see that $(H\rho_{R,S})^p(H\rho_{R,S})' = U_{R,S}$ and if $\alpha = \widehat{g}|_{H\rho_{R,S}}$ for some $g \in G_{R,S}$, then $\overline{\alpha} = \widehat{gU_{R,S}}|_{H\rho_{R,S}/U_{R,S}}$ (here the symbol $\widehat{x}$ denotes the inner automorphism given by x). Thus, it is sufficient to prove that $\mathrm{Aut}_{G_{R,S}/U_{R,S}}(H\rho_{R,S}/U_{R,S})$ is a p-group.

Let us note that $U(p)R = V(p)$ and $U(p)S = W(p)$. Indeed, it follows from the inclusion $L \leqslant U(p)$ that $R \leqslant V(p)$ and therefore $U(p)R \leqslant V(p)$. Since R is p'-isolated in A and, as proven above, HR/R is finite, the quotient group A/R is p'-torsion-free and the finite subgroup $U(p)R/R$ is p'-isolated in this group. Hence, the subgroup $U(p)R$ is p'-isolated in A. The last fact and the inclusion $U(p) \cdot \ker \lambda \leqslant U(p)R$ mean that $V(p) \leqslant U(p)R$. The equality $U(p)S = W(p)$ can be proved in the same way.

It is easy to see that $G_{R,S}/U_{R,S}$ is the generalized free product of the groups $A\rho_{R,S}/U_{R,S}$ and $B\rho_{R,S}/U_{R,S}$ with the subgroup $H\rho_{R,S}/U_{R,S}$ amalgamated. Since $\rho_{R,S}$ continues the natural homomorphisms $A \to A/R$ and $B \to B/S$, the following relations hold:

$$A\rho_{R,S}/U_{R,S} \cong (A/R)/(U(p)R/R) \cong A/V(p),$$
$$B\rho_{R,S}/U_{R,S} \cong (B/S)/(U(p)S/S) \cong B/W(p).$$

The indicated isomorphisms define an isomorphism of $G_{R,S}/U_{R,S}$ onto $G_{V(p),W(p)}$, which maps $H\rho_{R,S}/U_{R,S}$ onto $H\rho_{V(p),W(p)}$ and thus induces an isomorphism of the groups

$$\mathrm{Aut}_{G_{R,S}/U_{R,S}}(H\rho_{R,S}/U_{R,S}) \quad \text{and} \quad \mathrm{Aut}_{G_{V(p),W(p)}}(H\rho_{V(p),W(p)}).$$

By the condition of the proposition, the latter is a p-group, as required.

So, $\mathrm{Aut}_{G_{R,S}}(H\rho_{R,S}) \in \mathcal{F}_p$ and $H\rho_{R,S}$ is finite. Due to Propositions 5.4 and 3.1, it follows that $G_{R,S}$ is residually an $\mathcal{F}_p$-group and there exists a subgroup $N_{R,S} \in \mathcal{F}_p^*(G_{R,S})$ satisfying the relation $N_{R,S} \cap H\rho_{R,S} = 1$. If N denotes the pre-image of $N_{R,S}$ under $\rho_{R,S}$, then $N \in \mathcal{F}_p^*(G)$ and $N \cap H \leqslant \ker \rho_{R,S} \cap H = R \cap H$ because $\rho_{R,S}$ continues the natural homomorphism $A \to A/R$. Since $R \cap H = L$ and $L \leqslant M$ by the definition of L, it follows that N is the desired subgroup.

8. Main theorem

As noted in Section 2, the next theorem serves as the main tool that allows one to use results on the residual p-finiteness of the group $G = \langle A * B;\ H\rangle$ to study the residual nilpotence of this group.

Theorem 7. *Suppose that $G = \langle A * B;\ H\rangle$ and $\mathfrak{P}$ is a non-empty set of primes. Suppose also that the subgroups 1 and H are $\mathcal{FN}_{\mathfrak{P}}$-separable in each of the free factors and, for any $p \in \mathfrak{P}$, G is $\mathcal{F}_p$-quasi-regular with respect to H. Then the following statements hold.*

1. *The group G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and residually an $\mathcal{F}_p \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each prime p.*

2. *If, for some $q \in \mathfrak{P}$, the set $U = q'\text{-}\mathfrak{Rt}(H, 1)$ is a subgroup, which is $\mathcal{F}_q$-separable in H, and H is $\mathcal{F}_q$-separable in each of the free factors, then G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group.*

Proof. By Proposition 7.1, for any $p \in \mathfrak{P}$, the $\mathcal{F}_p$-quasi-regularity of G with respect to H implies the $\mathcal{F}_p$-quasi-regularity of this group with respect to A and B. Let us show that Conditions (α)—(γ) from Proposition 5.2 hold for the group G and the class $\mathcal{C} = \mathcal{FN}_{\mathfrak{P}}$.

Indeed, if $X, Y \in \mathcal{FN}^*_{\mathfrak{P}}(G)$ and $Z = X \cap Y$, then $Z \in \mathcal{FN}^*_{\mathfrak{P}}(G)$ by Proposition 3.1. Hence, (γ) holds. Suppose that $K = 1$ or $K = H$, and $a \in A \setminus K$. By the condition of the theorem, K is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and therefore $a \notin KM$ for some subgroup $M \in \mathcal{FN}^*_{\mathfrak{P}}(A)$. Due to Proposition 4.1, the $\mathcal{FN}_{\mathfrak{P}}$-group A/M can be decomposed into the direct product of its Sylow subgroups T_1/M, T_2/M, $\ldots$, T_n/M. If M_i, $1 \leqslant i \leqslant n$, denotes the subgroup $\prod_{j \neq i} T_j$, then $\bigcap_{i=1}^n M_i/M = 1$ and therefore $\bigcap_{i=1}^n M_i = M$. It is clear also that if the subgroup T_i/M corresponds to a number $p_i \in \mathfrak{P}$, then $M_i \in \mathcal{F}^*_{p_i}(A)$. Hence, it follows from the $\mathcal{F}_{p_i}$-quasi-regularity of G with respect to A that there exists a subgroup $N_i \in \mathcal{F}^*_{p_i}(G)$ satisfying the relation $N_i \cap A \leqslant M_i$. If $N = \bigcap_{i=1}^n N_i$, then $N \cap A \leqslant \bigcap_{i=1}^n M_i = M$, $a \notin K(N \cap A)$, and, by Proposition 3.1, $N \in \mathcal{FN}^*_{\mathfrak{P}}(G)$ because $\mathcal{F}^*_{p_i}(G) \subseteq \mathcal{FN}^*_{\mathfrak{P}}(G)$ for any $i \in \{1, 2, \ldots, n\}$. Since the element a is chosen arbitrarily, we have

$$\bigcap_{X \in \mathcal{FN}^*_{\mathfrak{P}}(G)} K(X \cap A) = K.$$

A similar argument can be used to prove the equalities from Conditions (α) and (β) of Proposition 5.2, which relate to the group B. Thus, all the conditions of the indicated proposition hold.

By Propositions 5.2 and 3.8, G is residually a $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group and any $\Phi \cdot \mathcal{FN}_{\mathfrak{P}}$-group is residually an $\mathcal{F}_p \cdot \mathcal{FN}_{\mathfrak{P}}$-group for every $p \in \mathfrak{P}$. Hence, Statement 1 of the theorem holds. Let us prove Statement 2.

Due to Proposition 3.2, the $\mathcal{F}_q$-separability of H in A and B implies that H is q'-isolated in these groups. Therefore, $q'\text{-}\mathfrak{Rt}(A, 1) = U = q'\text{-}\mathfrak{Rt}(B, 1)$. It follows that U is normal in A, in B, and hence in G. Let us show that this subgroup is $\mathcal{F}_q$-separable in A and B, H/U is $\mathcal{F}_q$-separable in A/U and B/U, and G/U is $\mathcal{F}_q$-quasi-regular with respect to A/U and B/U.

Indeed, let $a \in A \setminus U$. To prove the $\mathcal{F}_q$-separability of U in A, it is sufficient to find a subgroup $X \in \mathcal{F}^*_q(A)$ satisfying the relation $a \notin UX$. Since H is $\mathcal{F}_q$-separable in A, if $a \notin H$, there exists a subgroup $L \in \mathcal{F}^*_q(A)$ such that $a \notin HL$. Then $a \notin UL$ and therefore L is the desired subgroup. Let $a \in H$. It follows from the $\mathcal{F}_q$-separability of U in H that $a \notin UM$ for some subgroup $M \in \mathcal{F}^*_q(H)$. Since G is $\mathcal{F}_q$-quasi-regular with respect to H, there exists a subgroup $N \in \mathcal{F}^*_q(G)$ satisfying the condition $N \cap H \leqslant M$. If $a \in U(N \cap A)$, it follows from the inclusions $a \in H$ and $U \leqslant H$ that $a \in U(N \cap H) \leqslant UM$, in contradiction with the choice of M. Hence, $a \notin U(N \cap A)$, and because $N \cap A \in \mathcal{F}^*_q(A)$ by Proposition 3.1, $N \cap A$ is the desired subgroup.

The $\mathcal{F}_q$-separability of H/U in A/U is proved similarly. Namely, if $aU \in (A/U) \setminus (H/U)$, then $a \in A \setminus H$ and $a \notin HL$ for some subgroup $L \in \mathcal{F}^*_q(A)$ since H is $\mathcal{F}_q$-separable in A. It easily follows that $LU/U \in \mathcal{F}_q^*(A/U)$ and $aU \notin (H/U)(LU/U)$. Therefore, H/U is $\mathcal{F}_q$-separable in A/U.

Finally, if $M/U \in \mathcal{F}_q^*(A/U)$, then $M \in \mathcal{F}_q^*(A)$ and because G is $\mathcal{F}_q$-quasi-regular with respect to A, there exists a subgroup $N \in \mathcal{F}_q^*(G)$ satisfying the condition $N \cap A \leqslant M$. It is obvious that N is q'-isolated in G. Hence, $U \leqslant N$ and $N/U \cap A/U \leqslant M/U$. Since $N/U \in \mathcal{F}_q^*(G/U)$, it follows that G/U is $\mathcal{F}_q$-quasi-regular with respect to A/U. The $\mathcal{F}_q$-separability of U and H/U in B and B/U, respectively, as well as the $\mathcal{F}_q$-quasi-regularity of G/U with respect to B/U can be proved in a similar way.

By Proposition 3.3, the $\mathcal{F}_q$-separability of U in A and B means that A/U and B/U are residually $\mathcal{F}_q$-groups. Therefore, it follows from Proposition 5.3 that G/U is residually an $\mathcal{F}_q$-group. Let us now choose an element $g \in G \setminus \{1\}$ and indicate a homomorphism of G onto an $\mathcal{FN}_{\mathfrak{P}}$-group that takes g to a non-trivial element.

If $g \notin U$, then $gU \neq 1$ and the natural homomorphism $G \to G/U$ can be extended to the desired one because G/U is residually an $\mathcal{F}_q$-group and $\mathcal{F}_q \subseteq \mathcal{FN}_{\mathfrak{P}}$. If $g \in U$, then the existence of the required homomorphism follows from Statement 1 of Proposition 5.2, which, as above, is applied to the group G and the class $\mathcal{C} = \mathcal{FN}_{\mathfrak{P}}$. Thus, G is residually an $\mathcal{FN}_{\mathfrak{P}}$-group.

Corollary 4. *Suppose that the group $G = \langle A * B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Suppose also that, for each $p \in \mathfrak{P}$, G is $\mathcal{F}_p$-quasi-regular with respect to H. Then Statements* 1 *and* 2 *of Theorem* 1 *hold.*

Proof. By Condition $(*)$, A and B are residually $\mathcal{FN}_{\mathfrak{P}}$-groups. This is equivalent to the $\mathcal{FN}_{\mathfrak{P}}$-separability of the trivial subgroup in these groups. Thus, Statement 2 of Theorem 1 immediately follows from Theorem 7. Let us prove Statement 1.

If $q \in \mathfrak{P}$, then $\mathcal{F}_q \subseteq \mathcal{FN}_{\mathfrak{P}}$ and the $\mathcal{F}_q$-separability of H in A and B means that H is $\mathcal{FN}_{\mathfrak{P}}$-separable in these groups. By Condition $(*)$, H can be embedded in a $\mathcal{BN}_{\mathfrak{P}}$-group. Since $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_q$, Propositions 4.2, 4.1, and 4.4 imply that $H \in \mathcal{BN}_q$, the set $U = q'\text{-}\mathfrak{Rt}(H, 1)$ is a subgroup, and this subgroup is $\mathcal{F}_q$-separable in H. Thus, Statement 1 of Theorem 1 follows from Statement 2 of Theorem 7.

9. Proof of Theorems 1—4 and 6

Proposition 9.1. *Suppose that the group $G = \langle A*B;\ H\rangle$ and a set of primes $\mathfrak{P}$ satisfy $(*)$. Then H is $\mathfrak{P}'$-isolated in A and B if and only if it is $\mathcal{FN}_{\mathfrak{P}}$-separable in these groups. In particular, if H is periodic, then it is $\mathcal{FN}_{\mathfrak{P}}$-separable in A and B.*

Proof. Let λ and μ be homomorphisms which exist due to Condition $(*)$. By Proposition 4.1, the torsion subgroup of the nilpotent group $A\lambda$ can be decomposed into the direct product of the subgroups $\mathfrak{P}\text{-}\mathfrak{Rt}(A\lambda, 1)$ and $\mathfrak{P}'\text{-}\mathfrak{Rt}(A\lambda, 1)$, normal in $A\lambda$. Since A is residually an $\mathcal{FN}_{\mathfrak{P}}$-group, it follows from Proposition 3.2 that both A and H have no $\mathfrak{P}'$-torsion. The injectivity of λ on H implies that $H\lambda \cap \mathfrak{P}'\text{-}\mathfrak{Rt}(A\lambda, 1) = 1$. Therefore, the composition of λ and the natural homomorphism $A\lambda \to A\lambda/\mathfrak{P}'\text{-}\mathfrak{Rt}(A\lambda, 1)$ still acts injectively on H. By Proposition 4.2, the quotient group $A\lambda/\mathfrak{P}'\text{-}\mathfrak{Rt}(A\lambda, 1)$ belongs to the class $\mathcal{BN}_{\mathfrak{P}}^{\mathrm{tf}}$ of all $\mathfrak{P}'$-torsion-free $\mathcal{BN}_{\mathfrak{P}}$-groups. Since $\mathcal{FN}_{\mathfrak{P}} \subseteq \mathcal{BN}_{\mathfrak{P}}^{\mathrm{tf}}$, A is residually a $\mathcal{BN}_{\mathfrak{P}}^{\mathrm{tf}}$-group. Now it follows from Proposition 4.4 that if H is $\mathfrak{P}'$-isolated in A, then it is $\mathcal{FN}_{\mathfrak{P}}$-separable in this group. The converse statement is implied by Proposition 3.2. Similarly, it can be proved that H is $\mathfrak{P}'$-isolated in B if and only if it is $\mathcal{FN}_{\mathfrak{P}}$-separable in this group.

Assume now that H is a periodic group. If an element $a \in A$ and a number $q \in \mathfrak{P}'$ are such that $a^q \in H$, then the order r of a is finite. Since A is $\mathfrak{P}'$-torsion-free, r and q are co-prime. Hence, $a \in H$ and H is $\mathfrak{P}'$-isolated in A. As above, it follows that H is $\mathcal{FN}_{\mathfrak{P}}$-separable in A. The $\mathcal{FN}_{\mathfrak{P}}$-separability of H in B can be proved similarly.

Proposition 9.2. *Suppose that $G = \langle A * B;\ H\rangle$, $\mathfrak{P}$ is a non-empty set of primes, H is periodic and can be embedded in a $\mathcal{BN}_{\mathfrak{P}}$-group. If G is residually an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group for each $q \in \mathfrak{P}$, then, for any $p \in \mathfrak{P}$, there exist sequences of subgroups described in Statement* 1 *of Theorem* 3*.*

Proof. Let $p \in \mathfrak{P}$, and let $H(p) = p\text{-}\mathfrak{Is}(H, 1)$. Since H can be embedded in a $\mathcal{BN}_{\mathfrak{P}}$-group, it follows from Propositions 4.1 and 4.2 that $H(p) = p\text{-}\mathfrak{Rt}(H, 1)$, $H \in \mathcal{BN}_{\mathfrak{P}}$, and $H(p) \in \mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$. By Proposition 4.5, the p'-torsion-free periodic $\mathcal{BN}_p$-group $H(p)$ is finite. Let us show that there exists a homomorphism of G onto an $\mathcal{F}_p$-group which acts injectively on $H(p)$.

Indeed, if $\mathfrak{P} = \{p\}$, then $\mathcal{F}_p \cdot \mathcal{FN}_{\mathfrak{P}} = \mathcal{F}_p$, G is residually an $\mathcal{F}_p$-group, and the existence of the required homomorphism follows from Proposition 3.1. Therefore, we can assume further that $\mathfrak{P}$ contains a number q which is not equal to p. Since G is residually an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group and the class $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$ is closed under taking subgroups and direct products of a finite number of factors, Proposition 3.1 implies that G has a homomorphism σ onto an $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$-group which acts injectively on $H(p)$. By the definition of the class $\mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$, $G\sigma$ contains an $\mathcal{F}_q$-subgroup $M \in \mathcal{FN}^*_{\mathfrak{P}}(G\sigma)$. Let ε denote the natural homomorphism $G\sigma \to G\sigma/M$. Due to Proposition 4.1, the $\mathcal{FN}_{\mathfrak{P}}$-group $G\sigma\varepsilon$ can be decomposed into the direct product of the subgroups $p\text{-}\mathfrak{Rt}(G\sigma\varepsilon, 1) \in \mathcal{F}_p$ and $p'\text{-}\mathfrak{Rt}(G\sigma\varepsilon, 1)$. Since $H(p)$ is a p-group and $q \neq p$, the following relations hold:

$$H(p)\sigma \cap M = 1 = H(p)\sigma\varepsilon \cap p'\text{-}\mathfrak{Rt}(G\sigma\varepsilon, 1).$$

Hence, the composition of σ, ε, and the natural homomorphism

$$G\sigma\varepsilon \to G\sigma\varepsilon/(p'\text{-}\mathfrak{Rt}(G\sigma\varepsilon, 1))$$

is the desired map.

So, there exists a subgroup $N \in \mathcal{F}^*_p(G)$ that meets $H(p)$ trivially. Let

$$1 = X_0/N \leqslant X_1/N \leqslant \ldots \leqslant X_m/N = G/N$$

be a normal series of the $\mathcal{F}_p$-group G/N with factors of order p. Suppose also that the series

$$1 = G_0/N \leqslant G_1/N \leqslant \ldots \leqslant G_n/N = G/N$$

is obtained from the previous one by removing some terms in such a way that there are no duplicates among the subgroups $G_i \cap H(p)$, $0 \leqslant i \leqslant n$. Then

$$|G_{i+1} \cap H(p)/G_i \cap H(p)| = p, \quad 0 \leqslant i \leqslant n-1,$$

and, by Proposition 3.1, the subgroups $Q_i = G_i \cap H(p)$, $R_i = G_i \cap A$, and $S_i = G_i \cap B$, $0 \leqslant i \leqslant n$, are the desired ones.

Proof of Theorems 1, 3, and 6. Here, by Statements 1—3 of Theorem 6 we mean Statements 1—3 of Theorem 2. Theorem 5 and Propositions 9.1, 6.5 imply

– Statement 2 of Theorem 6 and the necessity of the conditions of Statement 3 of the same theorem;

– the $\mathcal{FN}_{\mathfrak{P}}$-separability of H in A and B, when the conditions of Theorem 3 hold;

– the property of H to be q'-isolated, which is contained in Statement 2 of Theorem 3.

Since $\mathcal{FN}_{\mathfrak{P}} \subseteq \mathcal{F}_q \cdot \mathcal{FN}_{\mathfrak{P}}$ for any $q \in \mathfrak{P}$, Proposition 9.2 completes the proof of Statement 2 of Theorem 3 and the necessity of the conditions of Statement 3 of the same theorem. Statements 1 and 2 of Theorem 1, Statements 1 of Theorems 3 and 6, and also the sufficiency of the conditions of Statements 3 of Theorems 3 and 6 follow from Proposition 7.5 and Corollary 4.

Proof of Theorem 2. Let λ and μ be homomorphisms which exist due to Condition $(*)$. Suppose also that $p \in \mathfrak{P}$ and $U(p)$, $V(p)$, and $W(p)$ are the subgroups defined in the same way as in Theorem 6. Denote by σ the mapping of $G/U(p)$ to $G_{V(p),W(p)}$ which takes a coset $gU(p)$ to the element $g\rho_{V(p),W(p)}$, where $g \in G$. It follows from the relations $U(p) \leqslant V(p) \cap W(p) \leqslant \ker \rho_{V(p),W(p)}$ that this mapping is correctly defined. It is also easy to see that σ is a surjective homomorphism. Since

$$(H/U(p))\sigma = H\rho_{V(p),W(p)}, \quad (A/U(p))\sigma = A\rho_{V(p),W(p)}, \quad \text{and}$$
$$(B/U(p))\sigma = B\rho_{V(p),W(p)},$$

it induces a homomorphism of the group $\mathfrak{G}(p) = \mathrm{Aut}_{G/U(p)}(H/U(p))$ onto the group

$$\overline{\mathfrak{G}}(p) = \mathrm{Aut}_{G_{V(p),W(p)}}(H\rho_{V(p),W(p)}),$$

which maps the subgroups

$$\mathfrak{A}(p) = \mathrm{Aut}_{A/U(p)}(H/U(p)) \quad \text{and} \quad \mathfrak{B}(p) = \mathrm{Aut}_{B/U(p)}(H/U(p))$$

onto the subgroups

$$\overline{\mathfrak{A}}(p) = \mathrm{Aut}_{A\rho_{V(p),W(p)}}(H\rho_{V(p),W(p)}) \quad \text{and} \quad \overline{\mathfrak{B}}(p) = \mathrm{Aut}_{B\rho_{V(p),W(p)}}(H\rho_{V(p),W(p)}),$$

respectively.

Let ε denote the natural homomorphism $A \to A/V(p)$. Since $\rho_{V(p),W(p)}$ extends ε, we have $\overline{\mathfrak{A}}(p) \cong \mathrm{Aut}_{A\varepsilon}(H\varepsilon)$. It follows from Proposition 4.2 and the inclusions $A\lambda \in \mathcal{BN}_{\mathfrak{P}}$ and $\ker\lambda \leqslant V(p)$ that $A\varepsilon \in \mathcal{BN}_{\mathfrak{P}}$. Obviously, $A\varepsilon$ is a p'-torsion-free group, which belongs to $\mathcal{BN}_p$ because $p \in \mathfrak{P}$. Hence, it is residually an $\mathcal{F}_p$-group by Proposition 4.4, and $H\varepsilon \in \mathcal{BN}_p$ by Proposition 4.2. Since $U(p)$ is p'-isolated in H, Proposition 5.6 implies that $H \cap V(p) = U(p)$ and $H\varepsilon \cong H/H \cap V(p) = H/U(p)$. Therefore, $H\varepsilon$ is a periodic p'-torsion-free $\mathcal{BN}_p$-group, which is finite by Proposition 4.5. Now it follows from Proposition 3.4 that $\mathrm{Aut}_{A\varepsilon}(H\varepsilon) \in \mathcal{F}_p$. The inclusion $\overline{\mathfrak{B}}(p) \in \mathcal{F}_p$ can be proved in a similar way.

It is clear that the group $\overline{\mathfrak{G}}(p)$ is generated by its subgroups $\overline{\mathfrak{A}}(p)$ and $\overline{\mathfrak{B}}(p)$. Moreover, when abelian, it coincides with the product of these subgroups. Therefore, any of Conditions (α)—(γ) implies that $\overline{\mathfrak{G}}(p) \in \mathcal{F}_p$. Thus, we can use Theorem 6, which says that Statements 1—3 hold.

Proof of Theorem 4. The $\mathcal{FN}_{\mathfrak{P}}$-separability of H in A and B follows from Proposition 9.1. Let us fix a number $p \in \mathfrak{P}$ and put $H(p) = p\text{-}\mathfrak{Is}(H, 1)$.

It is clear that the set $A(p) = p\text{-}\mathfrak{Rt}(A, 1)$ is invariant under any automorphism of A. Since the subgroup $\mathrm{sgp}\{\tau A\}$ can be embedded into a $\mathcal{BN}_{\mathfrak{P}}$-group and, in particular, is nilpotent, Proposition 4.1 and the equality $A(p) = p\text{-}\mathfrak{Rt}(\mathrm{sgp}\{\tau A\}, 1)$ imply that $A(p)$ is a subgroup. The obvious relation $p\text{-}\mathfrak{Rt}(H, 1) = A(p) \cap H$ means that the set $p\text{-}\mathfrak{Rt}(H, 1)$ is also a subgroup and hence coincides with $H(p)$. Since $A(p) \leqslant \mathrm{sgp}\{\tau A\}$ and $\mathcal{BN}_{\mathfrak{P}} \subseteq \mathcal{BN}_p$, it follows from Propositions 4.2, 4.5, and 4.3 that $A(p) \in \mathcal{BN}_p$, $A(p)$ is finite, and A is $\mathcal{F}_p$-quasi-regular with respect to $A(p)$. The last fact and the inclusion $1 \in \mathcal{F}_p^*(A(p))$ imply the existence of a subgroup $C \in \mathcal{F}_p^*(A)$ satisfying the relation $C \cap A(p) = 1$. A similar argument allows one to assert that $B(p)$ is a finite normal subgroup of B and $D \cap B(p) = 1$ for some subgroup $D \in \mathcal{F}_p^*(B)$. Since p is chosen arbitrarily and Theorem 3 is already proved, it remains to show that, for this p, the sequences of subgroups described in Statement 1 of Theorem 3 and the normal series from Statement 1 of the theorem to be proved exist simultaneously. Let

$$1 = Q_0 \leqslant Q_1 \leqslant \ldots \leqslant Q_n = H(p), \quad R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A, \quad \text{and}$$
$$S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B$$

be the sequences from Theorem 3. It is clear that

$$H(p) = p\text{-}\mathfrak{Rt}(H, 1) \leqslant A(p) \cap B(p),$$
$$(R_0 \cap C) \cap H(p) \leqslant R_0 \cap C \cap A(p) = 1 = R_0 \cap H(p),$$
$$(S_0 \cap D) \cap H(p) = S_0 \cap D \cap B(p) = 1 = S_0 \cap H(p).$$

By Proposition 3.1, $R_0 \cap C \in \mathcal{F}_p^*(A)$ and $S_0 \cap D \in \mathcal{F}_p^*(B)$. Therefore, we can further assume without loss of generality that $R_0 \leqslant C$, $S_0 \leqslant D$, and hence $R_0 \cap A(p) = 1 = S_0 \cap B(p)$.

Since any finite p-group has a normal series with factors of order p, the sequences

$$R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A \quad \text{and} \quad S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B \tag{3}$$

can be refined in such a way that the orders of their factors are also equal to p. Let

$$R_0 = M_0 \leqslant M_1 \leqslant \ldots \leqslant M_r = A \quad \text{and} \quad S_0 = N_0 \leqslant N_1 \leqslant \ldots \leqslant N_s = B$$

be the results of this refinement. If $K_i = M_i \cap A(p)$ and $L_j = N_j \cap B(p)$, where $0 \leqslant i \leqslant r$ and $0 \leqslant j \leqslant s$, then the members of the series

$$1 = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_r = A(p) \quad \text{and} \quad 1 = L_0 \leqslant L_1 \leqslant \ldots \leqslant L_s = B(p) \tag{4}$$

are normal in A and B, and their factors are of order 1 or p.

The equality $H(p) = p\text{-}\mathfrak{Rt}(H, 1)$ means that

$$A(p) \cap H = H(p) = B(p) \cap H, \quad K_i \cap H = M_i \cap H(p), \quad L_j \cap H = N_j \cap H(p),$$
$$0 \leqslant i \leqslant r, \quad 0 \leqslant j \leqslant s.$$

Since $R_i \cap H(p) = Q_i = S_i \cap H(p)$, $0 \leqslant i \leqslant n$, and $|Q_{i+1}/Q_i| = p$, $0 \leqslant i \leqslant n-1$, the members of any refinements of the sequences (3) meet $H(p)$ in the subgroups Q_i, $0 \leqslant i \leqslant n$. It follows that

$$\{K_i \cap H \mid 0 \leqslant i \leqslant r\} = \{M_i \cap H(p) \mid 0 \leqslant i \leqslant r\} = \{Q_i \mid 0 \leqslant i \leqslant n\} =$$
$$= \{N_j \cap H(p) \mid 0 \leqslant j \leqslant s\} = \{L_j \cap H \mid 0 \leqslant j \leqslant s\}.$$

Thus, all the conditions of Theorem 4 are satisfied for the series (4) if we remove duplicate members from them.

Now let

$$1 = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_k = A(p) \quad \text{and} \quad 1 = B_0 \leqslant B_1 \leqslant \ldots \leqslant B_m = B(p) \tag{5}$$

be the series described in Statement 1 of the theorem to be proved. Then

$$1 = A_0 \cap H(p) \leqslant A_1 \cap H(p) \leqslant \ldots \leqslant A_k \cap H(p) = H(p)$$

is a normal series of $H(p)$, whose factors are of order 1 or p. Since

$$\{A_i \cap H \mid 0 \leqslant i \leqslant k\} = \{B_j \cap H \mid 0 \leqslant j \leqslant m\},$$

the equality

$$\{A_i \cap H(p) \mid 0 \leqslant i \leqslant k\} = \{B_j \cap H(p) \mid 0 \leqslant j \leqslant m\}$$

holds. Hence, by removing certain members of (5), we can get normal series

$$1 = U_0 \leqslant U_1 \leqslant \ldots \leqslant U_n = A(p) \quad \text{and} \quad 1 = V_0 \leqslant V_1 \leqslant \ldots \leqslant V_n = B(p)$$

of the same length that satisfy the relations

$$U_i \cap H(p) = V_i \cap H(p), \quad 0 \leqslant i \leqslant n,$$
$$|U_{i+1} \cap H(p)/U_i \cap H(p)| = p, \quad 0 \leqslant i \leqslant n-1.$$

Let us put $R_i = U_iC$, $S_i = V_iD$, and $Q_i = U_i \cap H(p)$, where $0 \leqslant i \leqslant n$.

Since $C \in \mathcal{F}_p^*(A)$ and $D \in \mathcal{F}_p^*(B)$, we have $R_i \in \mathcal{F}_p^*(A)$ and $S_i \in \mathcal{F}_p^*(B)$. It follows from the equalities $C \cap A(p) = 1 = D \cap B(p)$ and the inclusions $U_i \leqslant A(p)$, $V_i \leqslant B(p)$, and $H(p) \leqslant A(p) \cap B(p)$ that

$$R_i \cap H(p) = U_i \cap H(p) = Q_i = V_i \cap H(p) = S_i \cap H(p), \quad 0 \leqslant i \leqslant n.$$

Thus, the sequences

$$1 = Q_0 \leqslant Q_1 \leqslant \ldots \leqslant Q_n = H(p), \quad R_0 \leqslant R_1 \leqslant \ldots \leqslant R_n = A, \quad \text{and}$$
$$S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = B$$

satisfy all the conditions of Theorem 3.

Data availability

No data was used for the research described in the article.

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