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On the conjugacy separability of ordinary and generalized Baumslag–Solitar groups ☆

E.V. Sokolov

Ivanovo State University, Ermak st., 39, 153025, Ivanovo, Russia

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ABSTRACT

Let $\mathcal{C}$ be a class of groups. A group X is said to be residually a $\mathcal{C}$-group (conjugacy $\mathcal{C}$-separable) if, for any elements $x, y \in X$ that are not equal (not conjugate in X), there exists a homomorphism σ of X onto a group from $\mathcal{C}$ such that the elements $x\sigma$ and $y\sigma$ are still not equal (respectively, not conjugate in $X\sigma$). A generalized Baumslag–Solitar group or GBS-group is the fundamental group of a finite connected graph of groups whose vertex and edge groups are all infinite cyclic. An ordinary Baumslag–Solitar group is the GBS-group that corresponds to a graph containing only one vertex and one loop. Suppose that the class $\mathcal{C}$ consists of periodic groups and is closed under taking subgroups and unrestricted wreath products. We prove that a non-solvable GBS-group is conjugacy $\mathcal{C}$-separable if and only if it is residually a $\mathcal{C}$-group. We also find a criterion for a solvable GBS-group to be conjugacy $\mathcal{C}$-separable. As a corollary, we prove that an arbitrary GBS-group is conjugacy (finite) separable if and only if it is residually finite.

1. Introduction. Statement of results

Recall that a *generalized Baumslag–Solitar group* (or, more briefly, a *GBS-group*) is the fundamental group of a finite connected graph of groups whose vertex and edge groups are all infinite cyclic (the definition of the fundamental group of a graph of groups can be found in Section 3). Although fundamental groups of this type have been studied since the early 1990s, the term "generalized Baumslag–Solitar group" came into use only in the 2000s (see [21–23,29]).

If the graph of groups contains only one vertex and one loop, then the corresponding GBS-group turns out to be an ordinary *Baumslag–Solitar group* [1], i.e., the group defined by the presentation of the form

$$\mathrm{BS}(m,n) = \langle a, t;\ t^{-1}a^m t = a^n \rangle,$$

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E-mail address: ev-sokolov@yandex.ru.

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where $m \neq 0 \neq n$. Since the groups $\mathrm{BS}(m,n)$, $\mathrm{BS}(n,m)$, $\mathrm{BS}(-m,-n)$, and $\mathrm{BS}(-n,-m)$ are pairwise isomorphic, we can assume without loss of generality that $0 < m \leqslant |n|$ in the above presentation.

A GBS-group is said to be *elementary* if it is isomorphic to an infinite cyclic group or the group $\mathrm{BS}(1,n)$, where $|n| = 1$ [30, p. 6]. It is known [7] that a GBS-group is solvable if and only if it is elementary or isomorphic to the group $\mathrm{BS}(1,n)$, where $|n| > 1$.

Generalized Baumslag–Solitar groups have been the subject of numerous investigations over the last thirty years (see, for example, [2,4–6,9–13,15–19,24,31,37]). In particular, residual properties of such groups are considered in [14,30,36,41]. In this article, we study the conjugacy separability of GBS-groups.

Suppose that $\mathcal{C}$ is a class of groups, X is a group, and θ is a relation between elements and (or) subsets of the group X, which is defined in each homomorphic image of X. The group X is said to be *residually a $\mathcal{C}$-group with respect to θ* if, for any elements and subsets of X that are not in θ, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ (a $\mathcal{C}$-group) such that the images of these elements and subsets under σ are still not in θ. If θ is the relation of equality of two elements, then it is usually not mentioned and X is simply called *residually a $\mathcal{C}$-group* [26]. In particular, if $\mathcal{C}$ coincides with the class $\mathcal{F}$ of all finite groups, then X is said to be *residually finite*. When θ is the relation of conjugacy of two elements, X is called a *conjugacy $\mathcal{C}$-separable group* [35] or simply a *conjugacy separable* one if $\mathcal{C} = \mathcal{F}$. Since equal elements are obviously conjugate, it follows from the conjugacy $\mathcal{C}$-separability of a group that the latter is residually a $\mathcal{C}$-group. The converse is not always true, and this fact is confirmed, in particular, by Theorem 2 given below (see the last paragraph of this section for details).

Let us say that a class of groups is *non-trivial* if it contains at least one non-trivial group. A non-trivial class of groups $\mathcal{C}$ is called a *root class* if it is closed under taking subgroups and satisfies any of the following three equivalent conditions (see [39]):

1)given a group X and a subnormal series $1 \leqslant Z \leqslant Y \leqslant X$, it follows from the inclusions $X/Y \in \mathcal{C}$ and $Y/Z \in \mathcal{C}$ that there exists a normal subgroup T of X such that $T \leqslant Z$ and $X/T \in \mathcal{C}$ (the Gruenberg condition [25]);

2) the class $\mathcal{C}$ is closed under taking unrestricted wreath products;

3) the class $\mathcal{C}$ is closed under taking extensions and unrestricted direct products of the form $\prod_{y \in Y} X_y$, where $X, Y \in \mathcal{C}$ and X_y is an isomorphic copy of X for each $y \in Y$.

Some examples of root classes are the class of all finite groups; the class of finite p-groups, where p is a prime; the class of periodic $\mathfrak{P}$-groups of finite exponent, where $\mathfrak{P}$ is a non-empty set of primes; the classes of all solvable groups and all torsion-free groups. It is easy to see that a non-trivial intersection of any number of root classes is again a root class. It is also clear that a non-trivial class of groups consisting only of finite groups is a root class if and only if it is closed under taking subgroups and extensions.

The use of the concept of a root class turned out to be very productive in the study of some residual properties of free constructions of groups: generalized free and tree products, HNN-extensions, fundamental groups of graphs of groups, etc. It allows one to prove several statements at once and quickly complicate the constructions under consideration; see, for example, [39,41–47]. At the same time, if $\mathcal{C}$ is a root class different from the class of all finite groups, very few facts are known about the conjugacy $\mathcal{C}$-separability of free constructions of groups. Most of the results of such type are contained in [3,20,27,28,34,40], and this article complements this list.

Let $\mathfrak{P}$ be a set of primes. Recall that an integer is said to be a *$\mathfrak{P}$-number* if all its prime divisors belong to $\mathfrak{P}$; a periodic group is said to be a *$\mathfrak{P}$-group* if the orders of all its elements are $\mathfrak{P}$-numbers. Given a class of groups $\mathcal{C}$ consisting only of periodic groups, let us denote by $\mathfrak{P}(\mathcal{C})$ the set of primes defined as follows: $p \in \mathfrak{P}(\mathcal{C})$ if and only if p divides the order of an element of some $\mathcal{C}$-group.

In [41,46], a criterion was found for a GBS-group to be residually a $\mathcal{C}$-group, where $\mathcal{C}$ was a root class of groups consisting only of periodic groups. The main result of this article is

Theorem 1. *Suppose that $\mathfrak{G}$ is a GBS-group, $\mathcal{C}$ is a root class of groups consisting only of periodic groups, and $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ is the class of all finite solvable $\mathfrak{P}(\mathcal{C})$-groups. Suppose also that the group $\mathfrak{G}$ is either elementary or non-solvable. Then the following statements are equivalent.*

1. *The group $\mathfrak{G}$ is residually a $\mathcal{C}$-group.*
2. *The group $\mathfrak{G}$ is conjugacy $\mathcal{C}$-separable.*
3. *The group $\mathfrak{G}$ is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable.*

Theorem 1 does not cover only the case when $\mathfrak{G}$ is isomorphic to the group $\mathrm{BS}(1, n)$ for some $n \in \mathbb{Z} \setminus \{0, \pm 1\}$. In this case, a criterion for $\mathfrak{G}$ to be conjugacy $\mathcal{C}$-separable is given by

Theorem 2. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups, $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ is the class of all finite solvable $\mathfrak{P}(\mathcal{C})$-groups, and $n \in \mathbb{Z} \setminus \{0, \pm 1\}$. Then the following statements are equivalent.*

1. *The group $\mathrm{BS}(1, n)$ is conjugacy $\mathcal{C}$-separable.*
2. *The group $\mathrm{BS}(1, n)$ is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable.*
3. *The set $\mathfrak{P}(\mathcal{C})$ contains all prime numbers.*

The formulated theorems imply

Corollary. *An arbitrary GBS-group is conjugacy separable if and only if it is residually finite.*

Theorems 1 and 2 generalize earlier results on the conjugacy separability of ordinary Baumslag–Solitar groups: Theorems 10–12 from [34] and the main result of [33]. At the same time, the proof of Theorem 2 essentially uses Theorem 10 from [34].

Let us make a remark regarding the criterion of conjugacy separability from Theorem 2. If $\mathcal{C}$ is a root class of groups consisting only of periodic groups and $n \in \mathbb{Z} \setminus \{0, \pm 1\}$, then the group $\mathrm{BS}(1, n)$ is residually a $\mathcal{C}$-group if and only if there exists a number $p \in \mathfrak{P}(\mathcal{C})$ such that p does not divide n and the order of n in the multiplicative group of the field $\mathbb{Z}_p$ is a $\mathfrak{P}(\mathcal{C})$-number [46]. In particular, $\mathrm{BS}(1, n)$ is residually a finite p-group for some prime p if and only if p divides $n - 1$ [34, Theorem 2]. It is also known that, for any prime p which does not divide $n - 1$, there exists a prime $q > p$ such that q does not divide $n - 1$ and $\mathrm{BS}(1, n)$ is residually an $\mathcal{F}_{\{p,q\}}$-group, where $\mathcal{F}_{\{p,q\}}$ is the class of finite $\{p, q\}$-groups [34, Corollary 1]. Moreover, the probability is quite high that $\mathrm{BS}(1, n)$ is residually an $\mathcal{F}_{\{p,q\}}$-group for randomly chosen primes p and q that do not divide $n - 1$ and do not exceed some given number [48]. Thus, there are many examples where $\mathcal{C}$ is a root class of groups consisting only of periodic groups, $n \in \mathbb{Z} \setminus \{0, \pm 1\}$, and $\mathrm{BS}(1, n)$ is residually a $\mathcal{C}$-group, but is not conjugacy $\mathcal{C}$-separable.

2. Solvable GBS-groups. Proof of Theorem 2

In this article, we use the following notation:

$\langle x \rangle$	the cyclic group generated by an element x;
(x, y)	the greatest common divisor of numbers x and y;
$[x, y]$	the commutator of elements x and y, which is equal to $x^{-1}y^{-1}xy$;
$x \mid y$	a number x divides a number y (in the sense defined below);
$x \sim_Z y$	elements x and y are conjugate in a group Z;
$\mathbb{Z}_{\geqslant 0}$	the set of non-negative integers;
$\mathcal{F}_{\mathfrak{P}}$	the class of all finite $\mathfrak{P}$-groups;
$\mathcal{FS}_{\mathfrak{P}}$	the class of all finite solvable $\mathfrak{P}$-groups.

Throughout the paper, we assume that, if $x, y \in \mathbb{Z}$, then $y \mid x$ if and only if there exists a number $z \in \mathbb{Z}$ satisfying the equality $x = yz$. In particular, $0 \mid x$ if and only if $x = 0$.

For any $n \in \mathbb{Z} \setminus \{0\}$, let

$$\Omega(n) = \big\{(r, s) \in \mathbb{Z}^2 \mid r > 0,\ s > 0,\ n^r \equiv 1 \ (\mathrm{mod}\ s)\big\},$$

and if $(r, s) \in \Omega(n)$, let

$$H(n, r, s) = \langle t, a;\ t^{-1}at = a^n,\ t^r = 1,\ a^s = 1\rangle.$$

Proposition 1. *If $n \in \mathbb{Z} \setminus \{0\}$ and $(r, s) \in \Omega(n)$, then the following statements hold.*

1. *Any element of the group $\mathrm{BS}(1, n)$ can be written in the form $t^u a^w t^{-v}$ for suitable numbers $u, v \in \mathbb{Z}_{\geqslant 0}$ and $w \in \mathbb{Z}$. In particular, this element is conjugate to an element of the form $t^u a^v$, where $u, v \in \mathbb{Z}$. Since the presentation of $\mathrm{BS}(1, n)$ is a part of the presentation of $H(n, r, s)$, the same statements also hold for the group $H(n, r, s)$.*

2. *Given $u \in \mathbb{Z}_{\geqslant 0}$ and $v, w \in \mathbb{Z}$, the elements $t^u a^v$ and $t^u a^w$ are conjugate in $\mathrm{BS}(1, n)$ (in $H(n, r, s)$) if and only if there exist $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $n^u - 1 \mid vn^x - wn^y$ (respectively, $(n^u - 1,\ s) \mid vn^x - wn^y$).*

Proof. 1. It follows from the relation $t^{-1}at = a^n$ that $a^k t = ta^{nk}$ and $t^{-1}a^k = a^{nk}t^{-1}$ for any $k \in \mathbb{Z}$. Clearly, these equalities allow us to transform any element of the group $\mathrm{BS}(1, n)$ to the required form.

2. By Statement 1, if $X = \mathrm{BS}(1, n)$ or $X = H(n, r, s)$, then $t^u a^v \sim_X t^u a^w$ if and only if there exist $x, y \in \mathbb{Z}_{\geqslant 0}$ and $z \in \mathbb{Z}$ such that $(t^x a^z t^{-y})^{-1} t^u a^v (t^x a^z t^{-y}) = t^u a^w$. We have

$$\begin{aligned} t^y a^{-z} t^{-x} t^u a^v t^x a^z t^{-y} = t^u a^w &\Leftrightarrow t^u (t^{-u} a^{-z} t^u)(t^{-x} a^v t^x) a^z = t^u (t^{-y} a^w t^y) \\ &\Leftrightarrow a^{vn^x - z(n^u - 1)} = a^{wn^y} \\ &\Leftrightarrow \begin{cases} vn^x - (n^u - 1)z = wn^y & \text{if } X = \mathrm{BS}(1, n), \\ vn^x - (n^u - 1)z \equiv wn^y \ (\mathrm{mod}\ s) & \text{if } X = H(n, r, s). \end{cases} \end{aligned}$$

It remains to note that

$$\big(\exists z \in \mathbb{Z}\ [vn^x - (n^u - 1)z = wn^y]\big) \Leftrightarrow \big(n^u - 1 \mid vn^x - wn^y\big)$$

and, since $(n^u - 1,\ s) = (n^u - 1)p + sq$ for some $p, q \in \mathbb{Z}$,

$$\begin{aligned} \exists z \in \mathbb{Z}\ [vn^x - (n^u - 1)z \equiv wn^y \pmod{s}] &\Leftrightarrow \exists z, \ell \in \mathbb{Z}\ [vn^x - (n^u - 1)z = wn^y + s\ell] \\ &\Leftrightarrow \exists m \in \mathbb{Z}\ [vn^x - (n^u - 1,\ s)m = wn^y] \\ &\Leftrightarrow (n^u - 1,\ s) \mid vn^x - wn^y. \quad \square \end{aligned}$$

Given a number $n \in \mathbb{Z} \setminus \{0\}$ and a non-empty set of primes $\mathfrak{P}$, let $\Omega(n, \mathfrak{P})$ denote the subset of $\Omega(n)$ defined as follows: $(r, s) \in \Omega(n, \mathfrak{P})$ if and only if r and s are $\mathfrak{P}$-numbers and $(r, s) \in \Omega(n)$. Let also

$$\Xi(n, \mathfrak{P}) = \big\{s \in \mathbb{Z} \mid \exists r \in \mathbb{Z}\ [(r, s) \in \Omega(n, \mathfrak{P})]\big\}.$$

Proposition 2. *If $n \in \mathbb{Z} \setminus \{0\}$ and $\mathfrak{P}$ is a non-empty set of primes, then the set $\Xi(n, \mathfrak{P})$ is closed under the operations of multiplication and taking a positive divisor.*

Proof. Let $s \in \Xi(n, \mathfrak{P})$. By the definition of $\Xi(n, \mathfrak{P})$, s is a positive $\mathfrak{P}$-number and there exists a positive $\mathfrak{P}$-number r such that $n^r \equiv 1 \pmod{s}$. Obviously, if $s' > 0$ and $s' \mid s$, then s' is a $\mathfrak{P}$-number and $n^r \equiv 1 \pmod{s'}$. Therefore, the set $\Xi(n, \mathfrak{P})$ is closed under taking positive divisors. Let us show by induction on k that $n^{rs^{k-1}} \equiv 1 \pmod{s^k}$ for any $k \geqslant 1$. Indeed, the basis of induction holds. If α_k denotes the number $n^{rs^{k-1}}$ and $\alpha_k \equiv 1 \pmod{s^k}$ for some $k \geqslant 1$, then

$$1 + \alpha_k + \alpha_k^2 + \ldots + \alpha_k^{s-1} \equiv s \pmod{s^k}.$$

It follows that $s \mid 1 + \alpha_k + \alpha_k^2 + \ldots + \alpha_k^{s-1}$ and, therefore,

$$s^{k+1} \mid (\alpha_k - 1)(1 + \alpha_k + \alpha_k^2 + \ldots + \alpha_k^{s-1}) = n^{rs^k} - 1.$$

Suppose now that r_1, r_2, s_1, and s_2 are positive $\mathfrak{P}$-numbers such that $n^{r_1} \equiv 1 \pmod{s_1}$ and $n^{r_2} \equiv 1 \pmod{s_2}$. Let d denote the greatest divisor of s_1 that is coprime with s_2, and let $\ell = s_1/d$. Then all prime divisors of ℓ divide s_2 and, therefore, $\ell \mid s_2^k$ for some $k \geqslant 1$. As proven above, $n^{r_2 s_2^k} \equiv 1 \pmod{s_2^{k+1}}$. The relations $n^{r_1} \equiv 1 \pmod{s_1}$ and $d \mid s_1$ mean that $n^{r_1} \equiv 1 \pmod{d}$. Hence, $n^{r_1 r_2 s_2^k} \equiv 1 \pmod{s_2^{k+1}}$, $n^{r_1 r_2 s_2^k} \equiv 1 \pmod{d}$, and $n^{r_1 r_2 s_2^k} \equiv 1 \pmod{d s_2^{k+1}}$ because $(d, s_2) = 1$. Since $s_1 s_2 = d\ell s_2 \mid d s_2^{k+1}$, it follows that $n^{r_1 r_2 s_2^k} \equiv 1 \pmod{s_1 s_2}$. It remains to note that $s_1 s_2$ and $r_1 r_2 s_2^k$ are, obviously, positive $\mathfrak{P}$-numbers and, therefore, $s_1 s_2 \in \Xi(n, \mathfrak{P})$. □

Proposition 3. *The following statements hold.*

1. *If $n \in \mathbb{Z} \setminus \{0\}$ and $(r, s) \in \Omega(n)$, then $H(n, r, s)$ is a finite solvable group of order rs.*
2. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and $n \in \mathbb{Z} \setminus \{0\}$. Then every homomorphism of* $\mathrm{BS}(1, n)$ *onto a group from $\mathcal{C}$ factors through the natural homomorphism $\eta \colon \mathrm{BS}(1, n) \to H(n, r, s)$ for some $(r, s) \in \Omega(n, \mathfrak{P}(\mathcal{C}))$. In particular, if the group* $\mathrm{BS}(1, n)$ *is conjugacy $\mathcal{C}$-separable, then it is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable.*

Proof. 1. It follows from the equalities $t^{-1}at = a^n$ and $tat^{-1} = t^{-(r-1)}at^{r-1} = a^{n^{r-1}}$ that the subgroup $\langle a \rangle$ is normal in the group $H(n, r, s)$ and, therefore, the latter is a split extension of this subgroup by the group $\langle t;\ t^r = 1 \rangle$. Let

$$\mathfrak{A} = \langle \mathfrak{a};\ \mathfrak{a}^s = 1 \rangle \quad \text{and} \quad \mathfrak{T} = \langle \mathfrak{t};\ \mathfrak{t}^r = 1 \rangle.$$

The inclusion $(r, s) \in \Omega(n)$ means that $n^r \equiv 1 \pmod{s}$ and, in particular, $(n, s) = 1$. Therefore, the endomorphism of $\mathfrak{A}$ taking $\mathfrak{a}$ to $\mathfrak{a}^n$ is an automorphism and its order divides r. This allows us to consider the split extension $\mathfrak{S}$ of $\mathfrak{A}$ by $\mathfrak{T}$ such that $\mathfrak{t}^{-1}\mathfrak{a}\mathfrak{t} = \mathfrak{a}^n$. Let ξ be the mapping of words in the alphabet $\{t, t^{-1}, a, a^{-1}\}$ that acts on the symbols of this alphabet in accordance with the rule

$$t \mapsto \mathfrak{t}, \quad t^{-1} \mapsto \mathfrak{t}^{-1}, \quad a \mapsto \mathfrak{a}, \quad a^{-1} \mapsto \mathfrak{a}^{-1}.$$

It is obvious that ξ takes all defining relations of $H(n, r, s)$ into the equalities valid in $\mathfrak{S}$ and, therefore, defines a homomorphism φ of the first group into the second one. Since $\langle a \rangle\varphi = \mathfrak{A}$, the order of the subgroup $\langle a \rangle$ equals s. Hence, the order of the group $H(n, r, s)$ is equal to rs.

2. Let σ be a homomorphism of $\mathrm{BS}(1, n)$ onto a $\mathcal{C}$-group X. Since $\mathcal{C}$ consists of periodic groups, the orders r and s of the elements $t\sigma$ and $a\sigma$ are finite and are $\mathfrak{P}(\mathcal{C})$-numbers. The equalities $(t\sigma)^{-1}a\sigma t\sigma = (a\sigma)^n$ and $(t\sigma)^r = 1$ imply that $a\sigma = (t\sigma)^{-r}a\sigma(t\sigma)^r = (a\sigma)^{n^r}$. Since the order of $a\sigma$ is equal to s, it follows that $n^r \equiv 1 \pmod{s}$. Thus, $(r, s) \in \Omega(n, \mathfrak{P}(\mathcal{C}))$ and we can consider the group $H(n, r, s)$. As above, if ξ is the mapping of words in the alphabet $\{t^{\pm 1}, a^{\pm 1}\}$ that acts on the symbols of this alphabet by the rule $t^{\pm 1} \mapsto (t\sigma)^{\pm 1}$, $a^{\pm 1} \mapsto (a\sigma)^{\pm 1}$, then ξ takes all defining relations of $H(n, r, s)$ into the equalities valid in

X and, therefore, defines a homomorphism $\rho\colon H(n,r,s) \to X$. By the construction, the homomorphisms σ and $\eta\rho$ act identically on the generators of $\mathrm{BS}(1,n)$. Hence, $\sigma = \eta\rho$ and $H(n,r,s)$ is the desired group. □

Proposition 4. [47, Proposition 8] *Let $\mathcal{C}$ be a root class of groups consisting only of periodic groups. A finite solvable group belongs to $\mathcal{C}$ if and only if it is a $\mathfrak{P}(\mathcal{C})$-group.*

Proposition 5. *If $m, n \in \mathbb{Z}$ and $0 < m \leqslant |n|$, then the following statements hold.*

1. *The group $\mathrm{BS}(m,n)$ is residually finite if and only if $m = 1$ or $m = |n|$.*
2. *If the group $\mathrm{BS}(m,n)$ is residually finite, then it is conjugacy separable.*
3. *Given a root class of groups $\mathcal{C}$ consisting only of periodic groups, the group $\mathrm{BS}(m,-m)$ is residually a $\mathcal{C}$-group if and only if m is a $\mathfrak{P}(\mathcal{C})$-number and $2 \in \mathfrak{P}(\mathcal{C})$.*

Proof. Statements 1 and 2 hold due to Theorem C from [32] and Theorem 10 from [34] respectively. Statement 3 is proved in [46] under the additional assumption that the class $\mathcal{C}$ is homomorphically closed. Proposition 2.3 from [41] states that this additional condition can be omitted. □

Proposition 6. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and $n \in \mathbb{Z} \setminus \{0\}$. Then the following statements hold.*

1. *The group $\mathrm{BS}(1,n)$ is conjugacy $\mathcal{C}$-separable if and only if*

$$\forall u \in \mathbb{Z}_{\geqslant 0}\ \forall v, w \in \mathbb{Z}\ \begin{bmatrix} \big(\forall x, y \in \mathbb{Z}_{\geqslant 0}\ [n^u - 1 \nmid vn^x - wn^y]\big) \Rightarrow \\ \big(\exists s \in \Xi(n, \mathfrak{P}(\mathcal{C}))\ \forall x, y \in \mathbb{Z}_{\geqslant 0}\ [(n^u - 1,\ s) \nmid vn^x - wn^y]\big) \end{bmatrix}. \tag{1}$$

2. *If the group $\mathrm{BS}(1,-1)$ is residually a $\mathcal{C}$-group, then it is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable.*

Proof. 1. Let

$$\mathcal{H} = \big\{H(n,r,s) \mid (r,s) \in \Omega(n, \mathfrak{P}(\mathcal{C}))\big\}.$$

It follows from Propositions 3 and 4 that $\mathcal{H} \subseteq \mathcal{FS}_{\mathfrak{P}(\mathcal{C})} \subseteq \mathcal{C}$ and the group $\mathrm{BS}(1,n)$ is conjugacy $\mathcal{C}$-separable if and only if it is conjugacy $\mathcal{H}$-separable. By Proposition 1, any element of $\mathrm{BS}(1,n)$ is conjugate to an element of the form $t^u a^v$, where $u, v \in \mathbb{Z}$. Suppose that $g = t^u a^v$, $g' = t^{u'} a^w$, $u \neq u'$, and r is a $\mathfrak{P}(\mathcal{C})$-number such that $r > |u| + |u'|$. Then $(r,1) \in \Omega(n, \mathfrak{P}(\mathcal{C}))$ and the natural homomorphism of $\mathrm{BS}(1,n)$ onto the finite cyclic group $H(n,r,1)$ takes g and g' to non-equal and, therefore, non-conjugate elements. Thus, $\mathrm{BS}(1,n)$ is conjugacy $\mathcal{H}$-separable if and only if

$$\forall u, v, w \in \mathbb{Z}\ \Big[\big(t^u a^v \nsim_{BS(1,n)} t^u a^w\big) \Rightarrow \big(\exists (r,s) \in \Omega(n, \mathfrak{P}(\mathcal{C}))\ [t^u a^v \nsim_{H(n,r,s)} t^u a^w]\big)\Big]. \tag{2}$$

Let X denote any of the groups $\mathrm{BS}(1,n)$ and $H(n,r,s)$. Then, if $u < 0$, the equalities

$$(t^u a^v)^{-1} = t^{-u} t^u a^{-v} t^{-u} = t^{-u} a^{-vn^{-u}}$$

hold in X, and $t^u a^v \sim_X t^u a^w$ if and only if $(t^u a^v)^{-1} \sim_X (t^u a^w)^{-1}$. Therefore, the number u in (2) can be considered non-negative. Since

$$\big(\exists (r,s) \in \Omega(n, \mathfrak{P}(\mathcal{C}))\big) \Leftrightarrow \big(\exists s \in \Xi(n, \mathfrak{P}(\mathcal{C}))\big),$$

it follows that (1) and (2) are equivalent due to Statement 2 of Proposition 1.

2. Let us show that if $n = -1$ and $\mathrm{BS}(1,-1)$ is residually a $\mathcal{C}$-group, then (2) holds and, therefore, the group $\mathrm{BS}(1,-1)$ is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable due to the above arguments.

Suppose that $u, v, w \in \mathbb{Z}$, $t^u a^v \nsim_{BS(1,-1)} t^u a^w$, and s is a $\{2\}$-number which is greater than $|v| + |w|$. Since $\mathrm{BS}(1,-1)$ is residually a $\mathcal{C}$-group, it follows from Statement 3 of Proposition 5 that $2 \in \mathfrak{P}(\mathcal{C})$ and, hence, $(2, s) \in \Omega(-1, \mathfrak{P}(\mathcal{C}))$. Since any element of the splitting extension $\mathrm{BS}(1,-1)$ can be written in the form $t^x a^y$ for some $x, y \in \mathbb{Z}$ and

$$(t^x a^y)^{-1} t^u a^v (t^x a^y) = t^u a^{(-1)^{|x|} v + (1-(-1)^{|u|}) y},$$

the conjugacy class Y of $t^u a^v$ in $\mathrm{BS}(1,-1)$ is either the set $\{t^u a^v, t^u a^{-v}\}$ (if u is even) or the set $\{t^u a^{v+2k} \mid k \in \mathbb{Z}\}$ (if u is odd). Clearly, if $\eta\colon \mathrm{BS}(1,-1) \to H(-1,2,s)$ is the natural homomorphism, then the conjugacy class of $t^u a^v = (t^u a^v)\eta$ in $H(-1,2,s)$ coincides with $Y\eta$. Since $|w \pm v| \leqslant |v| + |w| < s$ and s is even, it follows from the relation $t^u a^w \notin Y$ that $(t^u a^w)\eta \notin Y\eta$. Hence, $t^u a^v \nsim_{H(-1,2,s)} t^u a^w$ and (2) holds. $\square$

Proposition 7. *Suppose that $\mathcal{C}$ is a root class of groups consisting only of periodic groups and $\mathfrak{G}$ is a solvable GBS-group. Then the following statements are equivalent.*

1. *The group $\mathfrak{G}$ is conjugacy $\mathcal{C}$-separable.*
2. *The group $\mathfrak{G}$ is conjugacy $\mathcal{F}_{\mathfrak{P}(\mathcal{C})}$-separable.*
3. *The group $\mathfrak{G}$ is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable.*

Proof. The equivalence of Statements 2 and 3 follows from the fact that every homomorphic image of $\mathfrak{G}$ is a solvable group. The implication $3 \Rightarrow 1$ holds due to Proposition 4. The implication $1 \Rightarrow 3$ is obvious if $\mathfrak{G} \cong \mathbb{Z}$ and follows from Proposition 3 if $\mathfrak{G} \cong \mathrm{BS}(1,n)$ for some $n \in \mathbb{Z} \setminus \{0\}$. $\square$

Proof of Theorem 2. If the set $\mathfrak{P}(\mathcal{C})$ contains all primes, then the group $\mathrm{BS}(1,n)$ is conjugacy $\mathcal{F}_{\mathfrak{P}(\mathcal{C})}$-separable due to Statements 1 and 2 of Proposition 5. Therefore, the implication $3 \Rightarrow 2$, as well as the implication $2 \Rightarrow 1$, follows from Proposition 7. Let us show that if $u \geqslant 2$, $v = n^u - 1$, and $|v| \notin \Xi(n, \mathfrak{P}(\mathcal{C}))$, then (1) does not hold and, hence, the group $\mathrm{BS}(1,n)$ is not conjugacy $\mathcal{C}$-separable due to Proposition 6.

Indeed, since $|v| \notin \Xi(n, \mathfrak{P}(\mathcal{C}))$, $|n| \geqslant 2$, and $|v| \geqslant |n|^u - 1 \geqslant 3$, Proposition 2 implies the existence of a prime divisor q of v such that $q \notin \Xi(n, \mathfrak{P}(\mathcal{C}))$. Let $w = v/q$. It follows from the relations $v = n^u - 1$, $u > 0$, and $q \mid v$ that $q \nmid n$. Hence, $qw \nmid wn^y$ for any $y \in \mathbb{Z}_{\geqslant 0}$ and, therefore, $v = qw \nmid vn^x - wn^y$ for all $x, y \in \mathbb{Z}_{\geqslant 0}$. At the same time, if $s \in \Xi(n, \mathfrak{P}(\mathcal{C}))$, then $q \nmid s$ because the set $\Xi(n, \mathfrak{P}(\mathcal{C}))$ is closed under taking positive divisors. It follows that $(v, s) \mid w \mid vn^x - wn^y$ for any $x, y \in \mathbb{Z}_{\geqslant 0}$ and, thus, (1) does not hold.

Let us prove the implication $1 \Rightarrow 3$ by contradiction and assume that the group $\mathrm{BS}(1,n)$ is conjugacy $\mathcal{C}$-separable, but the set $\mathfrak{P}(\mathcal{C})$ does not contain some prime number p. If $u = p^2$ and $v = n^{p^2} - 1 = n^u - 1$, then $|v| \in \Xi(n, \mathfrak{P}(\mathcal{C}))$ as proven above. It follows from the definition of the set $\Xi(n, \mathfrak{P}(\mathcal{C}))$ that there exists a positive $\mathfrak{P}(\mathcal{C})$-number r such that $n^r \equiv 1 \pmod{|v|}$. Let us write r in the form $r = uk + \ell$, where $k \geqslant 0$ and $0 \leqslant \ell < u$. Then $1 \equiv n^r = n^{uk+\ell} \equiv n^\ell \pmod{|v|}$ because $n^u - 1 = v \equiv 0 \pmod{|v|}$. It follows from the inequalities $|n| \geqslant 2$, $\ell < u$, and $u \geqslant 4$ that

$$|n^\ell - 1| \leqslant |n|^\ell + 1 < |n|^{u-1} + (|n|^{u-1} - 1) \leqslant |n|^u - 1 \leqslant |v|$$

and, therefore, $n^\ell = 1$. Hence, $\ell = 0$, $p^2 = u \mid r$, and since r is a $\mathfrak{P}(\mathcal{C})$-number, $p \in \mathfrak{P}(\mathcal{C})$, which contradicts the assumption. $\square$

3. Graphs of groups and their fundamental groups

Suppose that Γ is a non-empty connected undirected graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ (loops and multiple edges are allowed). Let us associate each vertex $v \in \mathcal{V}$ with a group G_v, while each edge $e \in \mathcal{E}$

with a direction, a group H_e, and injective homomorphisms $\varphi_{+e}\colon H_e \to G_{e(1)}$ and $\varphi_{-e}\colon H_e \to G_{e(-1)}$, where $e(1)$ and $e(-1)$ are the vertices that are the ends of e. As a result, we get a directed *graph of groups*, which is further denoted by $\mathcal{G}(\Gamma)$. Let us refer to the groups G_v, the groups H_e, and the subgroups $H_{\varepsilon e} = H_e\varphi_{\varepsilon e}$, where $v \in \mathcal{V}$, $e \in \mathcal{E}$, and $\varepsilon = \pm 1$, as *vertex groups*, *edge groups*, and *edge subgroups* respectively.

The *fundamental group* of the graph $\mathcal{G}(\Gamma)$ is usually denoted by the symbol $\pi_1(\mathcal{G}(\Gamma))$ and can be defined as follows. Let $\mathcal{T}$ be a maximal subtree of Γ, and let $\mathcal{E}_\mathcal{T}$ be the edge set of $\mathcal{T}$. Then the generators of $\pi_1(\mathcal{G}(\Gamma))$ are the generators of the groups G_v, $v \in \mathcal{V}$, and symbols t_e, $e \in \mathcal{E} \setminus \mathcal{E}_\mathcal{T}$, while the defining relations are those of G_v, $v \in \mathcal{V}$, and also all possible relations of the forms

$$h_e\varphi_{+e} = h_e\varphi_{-e} \quad (e \in \mathcal{E}_\mathcal{T},\ h_e \in H_e),$$
$$t_e^{-1}(h_e\varphi_{+e})t_e = h_e\varphi_{-e} \quad (e \in \mathcal{E} \setminus \mathcal{E}_\mathcal{T},\ h_e \in H_e),$$

where $h\varphi_{\varepsilon e}$, $\varepsilon = \pm 1$, is a word in the generators of the group $G_{e(\varepsilon)}$ defining the image of h_e under $\varphi_{\varepsilon e}$ [38, §5.1].

Obviously, the presentation of the group $\pi_1(\mathcal{G}(\Gamma))$ depends on the choice of the maximal subtree $\mathcal{T}$. But it is known that all such presentations define isomorphic groups [38, §5.1], and this allows us to say about the fundamental group of a graph of groups without specifying the maximal subtree used to construct its presentation. It is also known that, for each vertex $v \in \mathcal{V}$, the group G_v is embedded into $\pi_1(\mathcal{G}(\Gamma))$ via the identical mapping of its generators and, therefore, can be considered a subgroup of $\pi_1(\mathcal{G}(\Gamma))$ [38, §5.2]. The following statement is a special case of Proposition 13 from [45].

Proposition 8. *Suppose that Γ is a finite graph and N is a normal subgroup of $\pi_1(\mathcal{G}(\Gamma))$. If $N \cap G_v = 1$ for every $v \in \mathcal{V}$, then N is a free group.*

If all vertex and edge groups of $\mathcal{G}(\Gamma)$ are infinite cyclic and we fix their generators g_v, $v \in \mathcal{V}$, and h_e, $e \in \mathcal{E}$, then, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, the homomorphism $\varphi_{\varepsilon e}$ is uniquely defined by the number $\lambda(\varepsilon e) \in \mathbb{Z}\setminus\{0\}$ such that $g_{e(\varepsilon)}^{\lambda(\varepsilon e)} = h_e\varphi_{\varepsilon e}$. Therefore, instead of $\mathcal{G}(\Gamma)$, we can consider the directed *graph with labels* $\mathcal{L}(\Gamma)$ which is obtained from Γ by choosing a direction for each edge $e \in \mathcal{E}$ and assigning non-zero integers $\lambda(+e)$ and $\lambda(-e)$ to the ends $e(1)$ and $e(-1)$ of this edge. If all vertex and edge groups of $\mathcal{G}(\Gamma)$ are finite cyclic, then $\mathcal{G}(\Gamma)$ can be replaced with the graph $\mathcal{M}(\Gamma)$, in which labels are assigned not only to the ends of the edges, but also to the vertices. Namely, the label $\mu(v)$ at a vertex v means that the group G_v is of order $\mu(v)$. Of course, the equality $|\mu(e(1))/\lambda(+e)| = |\mu(e(-1))/\lambda(-e)|$ must hold for each edge $e \in \mathcal{E}$. We call the group defined by the graph $\mathcal{L}(\Gamma)$ ($\mathcal{M}(\Gamma)$) the *fundamental group of the graph with labels* $\mathcal{L}(\Gamma)$ ($\mathcal{M}(\Gamma)$) and denote it by $\pi_1(\mathcal{L}(\Gamma))$ (respectively, $\pi_1(\mathcal{M}(\Gamma))$).

4. Non-solvable GBS-groups. Proof of Theorem 1

It follows from the last paragraph of Section 3 that each GBS-group can be defined by a graph with labels $\mathcal{L}(\Gamma)$ for some finite connected graph Γ and vice versa, each graph with labels $\mathcal{L}(\Gamma)$ over a non-empty finite connected graph Γ defines some GBS-group. Throughout this section, we assume that Γ is a non-empty finite connected graph with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$, $\mathcal{L}(\Gamma)$ is a graph with labels $\lambda(\varepsilon e)$, $e \in \mathcal{E}$, $\varepsilon = \pm 1$, and $\mathfrak{G}$ is the corresponding GBS-group with the vertex groups $G_v = \langle g_v \rangle$, $v \in \mathcal{V}$, and the edge subgroups $H_{\varepsilon e} = \langle g_{e(\varepsilon)}^{\lambda(\varepsilon e)} \rangle$, $e \in \mathcal{E}$, $\varepsilon = \pm 1$.

Suppose that $\mathcal{L}(\Gamma)$ contains an edge e such that e is not a loop and $|\lambda(\varepsilon e)| = 1$ for some $\varepsilon = \pm 1$. If we choose a maximal subtree of Γ containing e and use it to define a presentation of the group $\mathfrak{G}$, then the equality $g_{e(\varepsilon)} = g_{e(-\varepsilon)}^{\lambda(\varepsilon e)\lambda(-\varepsilon e)}$ holds in $\mathfrak{G}$ and, therefore, the generator $g_{e(\varepsilon)}$ can be excluded from this presentation. In $\mathcal{L}(\Gamma)$, this operation corresponds to the contraction of e with preliminary multiplication of all labels around the vertex $e(\varepsilon)$ by $\lambda(\varepsilon e)\lambda(-\varepsilon e)$. Such a transformation of $\mathcal{L}(\Gamma)$ is called an *elementary collapse* (see [29, p. 480]).

The graph $\mathcal{L}(\Gamma)$ is said to be *reduced* if, for any $e \in \mathcal{E}$, $\varepsilon = \pm 1$, it follows from the equality $|\lambda(\varepsilon e)| = 1$ that e is a loop [21, p. 224]. Since Γ is finite, we can always make $\mathcal{L}(\Gamma)$ reduced by using a finite number of elementary collapses.

If we replace the generator of some vertex group with its inverse, then all the labels around the corresponding vertex change sign. Similarly, replacing the generator of some edge group with its inverse affects the signs of the labels at the ends of this edge. The described replacements of the generators do not change the group $\mathfrak{G}$, and the corresponding graph transformations are called *admissible changes of signs* [29, p. 479].

Let a maximal subtree $\mathcal{T}$ of Γ be fixed. It is easy to see that, by applying suitable admissible changes of signs, we can make positive all the labels at the ends of the edges of $\mathcal{T}$. In this case, we say that the graph $\mathcal{L}(\Gamma)$ is *$\mathcal{T}$-positive.*

An element $a \in \mathfrak{G}$ is called an *elliptic* one if it is conjugate to an element of some vertex group. If the group $\mathfrak{G}$ is non-elementary, then any two elliptic elements $a, b \in \mathfrak{G} \setminus \{1\}$ are commensurable, i.e., $\langle a\rangle \cap \langle b\rangle \neq 1$ [29, Lemma 2.1]. This allows us to consider the mapping $\Delta\colon \mathfrak{G} \to \mathbb{Q}^*$, which is called the *modular homomorphism* of $\mathfrak{G}$ and is defined as follows.

Let $g \in \mathfrak{G}$. If a is a non-trivial elliptic element, then the element $g^{-1}ag$ is also elliptic and, therefore, $g^{-1}a^m g = a^n$ for some $m, n \in \mathbb{Z} \setminus \{0\}$. We put $\Delta(g) = n/m$. It can be proved that the number $\Delta(g)$ does not depend on the choice of a, m, and n; see [8, p. 1812].

The largest cyclic normal subgroup of $\mathfrak{G}$ is called the *cyclic radical* of this group and is denoted by $C(\mathfrak{G})$. The cyclic radical exists if $\mathfrak{G}$ is not isomorphic to $\mathrm{BS}(1,1)$ or $\mathrm{BS}(1,-1)$ [8, p. 1808].

Proposition 9. *Suppose that $\mathfrak{G}$ is non-elementary and $\operatorname{Im}\Delta \subseteq \{1,-1\}$. Suppose also that $\mathcal{T}$ is a maximal subtree of Γ used to define a presentation of $\mathfrak{G}$, $\mathcal{E}_\mathcal{T}$ is the edge set of $\mathcal{T}$, $\mathcal{L}(\Gamma)$ is reduced and $\mathcal{T}$-positive. Then the following statements hold.*

1. *The relations*

$$1 \neq C(\mathfrak{G}) = \bigcap_{\substack{e\in\mathcal{E},\\ \varepsilon=\pm1}} H_{\varepsilon e} \leqslant \bigcap_{v\in\mathcal{V}} G_v$$

hold, and therefore the number $\mu(v) = [G_v : C(\mathfrak{G})]$ is defined and finite for each $v \in \mathcal{V}$.

2. *The least common multiple μ of the numbers $\mu(v)$, $v \in \mathcal{V}$, divides $\prod_{e\in\mathcal{E},\, \varepsilon=\pm1} \lambda(\varepsilon e)$.*

3. *Suppose that Z is an infinite cyclic group with a generator z, A is the free abelian group with the basis $\{a_q \mid q \in \operatorname{Im}\Delta\}$, and X is the splitting extension of Z by A such that $a_q^{-1} z a_q = z^q$, $q \in \operatorname{Im}\Delta$. Then the mapping of the generators of $\mathfrak{G}$ into X given by the rule*

$$g_v \mapsto z^{\mu/\mu(v)}, \quad v \in \mathcal{V}, \quad t_e \mapsto a_{\Delta(t_e)}, \quad e \in \mathcal{E} \setminus \mathcal{E}_\mathcal{T},$$

defines a homomorphism $\tau\colon \mathfrak{G} \to X$.

4. *Suppose that $k \geqslant 1$, $Z_k = \langle z_k;\ z_k^{\mu k} = 1\rangle$, $B = \langle b;\ b^2 = 1\rangle$, $\eta_k\colon \mathfrak{G} \to \mathfrak{G}/(C(\mathfrak{G}))^k$ is the natural homomorphism, and X_k is either the group Z_k (if $\operatorname{Im}\Delta = \{1\}$) or the splitting extension of Z_k by B such that $b^{-1} z_k b = z_k^{-1}$ (if $\operatorname{Im}\Delta = \{1,-1\}$). Then*

a) *the map $\chi_k\colon X \to X_k$ given by the rule*

$$a_1^m z^\ell \mapsto z_k^\ell, \quad \ell, m \in \mathbb{Z}, \quad \textit{if } \operatorname{Im}\Delta = \{1\},$$
$$a_1^m a_{-1}^n z^\ell \mapsto b^n z_k^\ell, \quad \ell, m, n \in \mathbb{Z}, \quad \textit{if } \operatorname{Im}\Delta = \{1,-1\}$$

is a homomorphism;

b) *the map of the generators of* $\mathfrak{G}/(C(\mathfrak{G}))^k$ *into* X_k *given by the rule*

$$g_v \mapsto z_k^{\mu/\mu(v)}, \quad v \in \mathcal{V}, \quad t_e \mapsto \begin{cases} 1 & \text{if } \Delta(t_e) = 1, \\ b & \text{if } \Delta(t_e) = -1, \end{cases} \quad e \in \mathcal{E} \setminus \mathcal{E}_{\mathcal{T}},$$

defines a homomorphism $\tau_k \colon \mathfrak{G}/(C(\mathfrak{G}))^k \to X_k$;

c) *the equality* $\tau\chi_k = \eta_k\tau_k$ *holds*;

d) *the kernel of* τ_k *is a free group.*

Proof. Statements 1–3 are special cases of Propositions 4.4 and 5.1 from [41]. Let us prove Statement 4.

The fact that χ_k is a homomorphism can be verified directly. Let ξ_k be the mapping of words that continues the map from Statement 4-b. It is obvious that $\tau\chi_k(g) = \eta_k\xi_k(g)$ for each generator g of $\mathfrak{G}$. Therefore, the mapping ξ_k takes all defining relations of $\mathfrak{G}/(C(\mathfrak{G}))^k$ inherited from $\mathfrak{G}$ into the equalities valid in X_k. In addition to them, the presentation of the quotient group $\mathfrak{G}/(C(\mathfrak{G}))^k$ contains relations of the form $\omega = 1$, where ω is an arbitrary word that defines an element of $(C(\mathfrak{G}))^k$. Let $u \in \mathcal{V}$. Since $\mu(u) = [G_u : C(\mathfrak{G})]$, the subgroup $(C(\mathfrak{G}))^k$ is generated by the element $g_u^{\mu(u)k}$. Therefore, all these additional relations are derivable from the relation $g_u^{\mu(u)k} = 1$. It follows from the definitions of ξ_k and Z_k that ξ_k takes the latter relation into the equality valid in X_k. Thus, τ_k is a homomorphism and $\tau\chi_k = \eta_k\tau_k$.

It is obvious that the quotient group $\mathfrak{G}/(C(\mathfrak{G}))^k$ is the fundamental group of the graph with labels $\mathcal{M}(\Gamma)$ which is obtained from $\mathcal{L}(\Gamma)$ by assigning to each vertex $v \in \mathcal{V}$ the label $\mu(v)k$. Since the order of the group $Z_k^{\mu/\mu(v)} = \langle g_v \rangle\tau_k$ is also equal to $\mu(v)k$, the homomorphism τ_k acts injectively on all vertex groups. Now it follows from Proposition 8 that its kernel is a free group. □

Proposition 10. [41, Theorem 3] *Suppose that* $\mathcal{C}$ *is a root class of groups consisting only of periodic groups, the group* $\mathfrak{G}$ *is non-solvable, and the graph* $\mathcal{L}(\Gamma)$ *is reduced. Then the following statements hold.*

1. *If* $\operatorname{Im}\Delta = \{1\}$*, then* $\mathfrak{G}$ *is residually a* $\mathcal{C}$*-group if and only if all the labels of* $\mathcal{L}(\Gamma)$ *are* $\mathfrak{P}(\mathcal{C})$*-numbers.*
2. *If* $\operatorname{Im}\Delta = \{1, -1\}$*, then* $\mathfrak{G}$ *is residually a* $\mathcal{C}$*-group if and only if all the labels of* $\mathcal{L}(\Gamma)$ *are* $\mathfrak{P}(\mathcal{C})$*-numbers and* $2 \in \mathfrak{P}(\mathcal{C})$.
3. *If* $\operatorname{Im}\Delta \nsubseteq \{1, -1\}$*, then* $\mathfrak{G}$ *is not residually a* $\mathcal{C}$*-group.*

Proposition 11. [40, Theorem 1] *If* $\mathcal{C}$ *is a root class of groups consisting only of finite groups, then any extension of a free group by a* $\mathcal{C}$*-group is conjugacy* $\mathcal{C}$*-separable.*

Proposition 12. *Suppose that the group* $\mathfrak{G}$ *is non-solvable,* $\mathcal{T}$ *is a maximal subtree of* Γ *used to define a presentation of* $\mathfrak{G}$*, the graph* $\mathcal{L}(\Gamma)$ *is reduced and* $\mathcal{T}$*-positive. If* $\mathcal{C}$ *is a root class of groups consisting only of periodic groups and* $\mathfrak{G}$ *is residually a* $\mathcal{C}$*-group, then* $\mathfrak{G}$ *is conjugacy* $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$*-separable.*

Proof. It follows from Proposition 10 that $\operatorname{Im}\Delta \subseteq \{1, -1\}$, all the labels of $\mathcal{L}(\Gamma)$ are $\mathfrak{P}(\mathcal{C})$-numbers, and if $\operatorname{Im}\Delta = \{1, -1\}$, then $2 \in \mathfrak{P}(\mathcal{C})$. Given $k \geqslant 1$, let Z, Z_k, A, B, X, X_k and τ, τ_k, χ_k, η_k be the groups and the homomorphisms from Statements 3 and 4 of Proposition 9. By Statement 2 of the same proposition, μ is a $\mathfrak{P}(\mathcal{C})$-number. Therefore, $X_k \in \mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ if k is a $\mathfrak{P}(\mathcal{C})$-number.

Suppose that $f, g \in \mathfrak{G}$ and $f \nsim_{\mathfrak{G}} g$. Since the class $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ is closed under taking subgroups, to complete the proof, it suffices to find a homomorphism σ of $\mathfrak{G}$ into an $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-group $\overline{\mathfrak{G}}$ such that $f\sigma \nsim_{\overline{\mathfrak{G}}} g\sigma$. Let us consider two cases.

Case 1. $f \nsim_{\mathfrak{G}} g \pmod{(C(\mathfrak{G}))^k}$ for some $\mathfrak{P}(\mathcal{C})$-number $k \geqslant 1$.

By Statement 4-d of Proposition 9, the quotient group $\mathfrak{G}/(C(\mathfrak{G}))^k$ is an extension of a free group by the group $(\mathfrak{G}/(C(\mathfrak{G}))^k)\tau_k$. Since $(\mathfrak{G}/(C(\mathfrak{G}))^k)\tau_k \leqslant X_k \in \mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ and $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$ is a root class, the group

$\mathfrak{G}/(C(\mathfrak{G}))^k$ is conjugacy $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-separable due to Proposition 11. It follows from this fact and the relation $f\eta_k \nsim_{\mathfrak{G}\eta_k} g\eta_k$ that η_k can be continued to the desired homomorphism.

Case 2. $f \sim_{\mathfrak{G}} g \pmod{(C(\mathfrak{G}))^k}$ for any $\mathfrak{P}(\mathcal{C})$-number $k \geqslant 1$.

Since X is a splitting extension of Z by A, the element $g\tau$ can be written in the form $g\tau = az^n$ for suitable $a \in A$ and $n \in \mathbb{Z}$. Let m be a $\mathfrak{P}(\mathcal{C})$-number greater than $2|n|$. By the supposition, $f \sim_{\mathfrak{G}} g \pmod{(C(\mathfrak{G}))^m}$ and, therefore, $x^{-1}fx = gh$ for some $x \in \mathfrak{G}$ and $h \in (C(\mathfrak{G}))^m \setminus \{1\}$. Hence, we can assume further without loss of generality that $f = gh$. By the definition of τ, the equality $h\tau = z^{m\ell}$ holds for some $\ell \neq 0$. Let us choose a $\mathfrak{P}(\mathcal{C})$-number k greater than $2m|\ell|$ and show that $g\tau\chi_k \neq (gh)\tau\chi_k \neq g^{-1}\tau\chi_k$.

Indeed,

$$(g^2h)\tau = (az^n)(az^n)z^{m\ell} = a^2z^{n+\varepsilon n+m\ell},$$

where $\varepsilon = 1$ if a belongs to the centralizer of z in X and $\varepsilon = -1$ otherwise. By the definition of χ_k, the equalities $h\tau\chi_k = z_k^{m\ell}$ and $(g^2h)\tau\chi_k = z_k^{n+\varepsilon n+m\ell}$ hold. The relations $m > 2|n|$, $k > 2m|\ell|$, and $\ell \neq 0$ imply that $0 \neq |m\ell| < k/2 < \mu k$ and

$$0 < m - 2|n| \leqslant m|\ell| - 2|n| \leqslant |n + \varepsilon n + m\ell| \leqslant m|\ell| + 2|n| < m|\ell| + m \leqslant 2m|\ell| < k \leqslant \mu k.$$

Since the group Z_k is of order μk, it follows that $h\tau\chi_k \neq 1 \neq (g^2h)\tau\chi_k$ and, therefore, $g\tau\chi_k \neq (gh)\tau\chi_k \neq g^{-1}\tau\chi_k$ as required.

Let us now show that the conjugacy class $(g\tau\chi_k)^{X_k}$ of $g\tau\chi_k$ in X_k is contained in the set $\{g\tau\chi_k, g^{-1}\tau\chi_k\}$. Since $X_k \in \mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$, it will follow that $\tau\chi_k$ is the desired homomorphism.

If $\operatorname{Im}\Delta = \{1\}$, then X_k is an abelian group and, therefore, $(g\tau\chi_k)^{X_k} = \{g\tau\chi_k\}$. Suppose that $\operatorname{Im}\Delta = \{1, -1\}$. Let us fix a vertex $v \in \mathcal{V}$, denote by c the element $g_v^{\mu(v)}$ and assume that $[g, c] \neq 1$.

Since c generates the infinite cyclic subgroup $C(\mathfrak{G})$, which is normal in $\mathfrak{G}$, the equality $g^{-1}cg = c^{-1}$ holds. It follows that $g^{-1}c^{-1}g = c$, $c^{-1}gc = gc^2$, $cgc^{-1} = gc^{-2}$, and therefore $c^{-r}gc^r = gc^{2r}$ for any $r \in \mathbb{Z}$. The relation $\operatorname{Im}\Delta = \{1, -1\}$ implies that $2 \in \mathfrak{P}(\mathcal{C})$. Hence, $f \sim_{\mathfrak{G}} g \pmod{(C(\mathfrak{G}))^2}$ and $y^{-1}fy = gc^{2s}$ for some $y \in \mathfrak{G}$ and $s \in \mathbb{Z}$. Since $gc^{2s} = c^{-s}gc^s$, it follows that the elements f and g are conjugate in $\mathfrak{G}$, and we get a contradiction with their choice.

Thus, $[g, c] = 1$ and, therefore, the element $g\tau\chi_k$ belongs to the centralizer $\mathcal{Z}_{X_k}(c\tau\chi_k)$ of $c\tau\chi_k$ in X_k. Let us note that, if $u \in Z_k$, then $w^{-1}uw = u$ and $(bw)^{-1}u(bw) = u^{-1}$ for any $w \in Z_k$. Hence, the conjugacy class of u in X_k is the set $\{u, u^{-1}\}$, while the centralizer of u is either Z_k (if $u^2 \neq 1$) or X_k (otherwise). By the definitions of τ and χ_k, the equality $c\tau\chi_k = z_k^{\mu}$ holds. Since the group Z_k is of order μk and $k > 2m|\ell| > 2$, it follows that $(c\tau\chi_k)^2 \neq 1$. Hence, $g\tau\chi_k \in \mathcal{Z}_{X_k}(c\tau\chi_k) = Z_k$ and, therefore, $(g\tau\chi_k)^{X_k} = \{g\tau\chi_k, g^{-1}\tau\chi_k\}$ as required. $\square$

Proof of Theorem 1. The implication $2 \Rightarrow 1$ is obvious, the implication $3 \Rightarrow 2$ follows from Proposition 4. Let us prove the implication $1 \Rightarrow 3$.

By using elementary collapses and admissible changes of signs, we can transform $\mathcal{L}(\Gamma)$ at first to a reduced form, and then, after choosing some maximal subtree $\mathcal{T}$ of Γ, to a $\mathcal{T}$-positive form. Therefore, if $\mathfrak{G}$ is non-solvable, then the considered implication follows from Proposition 12. If $\mathfrak{G} \cong \mathrm{BS}(1, -1)$, the implication $1 \Rightarrow 3$ is ensured by Proposition 6.

Let $\mathfrak{G} \cong \mathbb{Z}$ or $\mathfrak{G} \cong \mathrm{BS}(1, 1) \cong \mathbb{Z} \times \mathbb{Z}$. Then $\mathfrak{G}$ is residually a $\mathcal{C}$-group if and only if it is conjugacy $\mathcal{C}$-separable because the relations of equality and conjugacy coincide for an abelian group. Since the image of any homomorphism of $\mathfrak{G}$ onto a group from $\mathcal{C}$ is, obviously, an $\mathcal{FS}_{\mathfrak{P}(\mathcal{C})}$-group, it follows that the implication $1 \Rightarrow 3$ also holds. $\square$

Declaration of competing interest

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