On the Subgroup Separability of the Free Product of Groups

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Abstract—Suppose that $\mathcal{C}$ is a root class of groups (i.e., a class of groups that contains non-trivial groups and is closed under taking subgroups and unrestricted wreath products), G is the free product of residually $\mathcal{C}$-groups A_i $(i \in \mathcal{I})$, and H is a subgroup of G satisfying a non-trivial identity. We prove a criterion for the $\mathcal{C}$-separability of H in G. It follows from this criterion that, if $\{\mathcal{V}_j | j \in \mathcal{J}\}$ is a family of group varieties, each $\mathcal{V}_j$ $(j \in \mathcal{J})$ is distinct from the variety of all groups, and $\mathcal{V} = \bigcup_{j \in \mathcal{J}} \mathcal{V}_j$, then one can give a description of $\mathcal{C}$-separable $\mathcal{V}$-subgroups of G provided such a description is known for every group A_i $(i \in \mathcal{I})$.

1. INTRODUCTION. STATEMENT OF RESULTS

Let $\mathcal{C}$ be a class of groups. Following [1], we say that a subgroup Y of a group X is $\mathcal{C}$-*separable* in this group if, for any element $x \in X \setminus Y$, there exists a homomorphism σ of X onto a group from $\mathcal{C}$ such that $x\sigma \notin Y\sigma$. Usually, the $\mathcal{C}$-separability of subgroups of X is studied under the assumption that X is *residually a $\mathcal{C}$-group*. Recall that the latter property is equivalent to the $\mathcal{C}$-separability of the trivial subgroup.

It is well known that, if the class $\mathcal{C}$ consists only of finite groups and the group X is finitely presented, then the $\mathcal{C}$-separability of the subgroup Y implies the existence of an algorithm that answers the question of whether a given element of X belongs to Y [1]. In addition, the $\mathcal{C}$-separability of certain subgroups of X can sometimes serve as one of the necessary and/or sufficient conditions for X to be residually a $\mathcal{C}$-group. This relationship is especially often found when X is a group-theoretic construction and $\mathcal{C}$ is a root class of groups (see, for example, [2–9]).

The notion of a root class was introduced in [10], and the equivalent definitions of such a class were given in [11]. In accordance with one of them, the class $\mathcal{C}$ is called a *root class* if it contains non-trivial groups and is closed under taking subgroups and unrestricted wreath products. Examples of root classes are the classes of all finite groups, finite p-groups (where p is a prime number), periodic $\mathfrak{P}$-groups of finite exponent (where $\mathfrak{P}$ is a non-empty set of primes), all solvable groups, and all torsion-free groups. It is also easy to show that, if the intersection of a family of root classes contains a non-trivial group, then it is again a root class.

Throughout the paper, let $\mathfrak{P}(\mathcal{C})$ denote the set of prime numbers defined as follows. If the class $\mathcal{C}$ contains a non-periodic group, then $\mathfrak{P}(\mathcal{C})$ is the set of all primes. Otherwise, a prime p belongs to $\mathfrak{P}(\mathcal{C})$ if and only if it divides the order of an element of some $\mathcal{C}$-group. Recall that the subgroup Y of the group X is said to be $\mathfrak{P}(\mathcal{C})'$-*isolated* in X if, for each element $x \in X$ and for each prime $q \notin \mathfrak{P}(\mathcal{C})$, the inclusion $x^q \in Y$ means that $x \in Y$. It is clear that, if $\mathfrak{P}(\mathcal{C})$ contains all prime numbers, then any subgroup is $\mathfrak{P}(\mathcal{C})'$-isolated.

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Proposition 2.2 below asserts that every $\mathcal{C}$-separable subgroup of X is $\mathfrak{P}(\mathcal{C})'$-isolated. If the group X is given by generators and defining relations, then the latter property is usually easier to verify than the $\mathcal{C}$-separability. Therefore, to get a criterion for the $\mathcal{C}$-separability of subgroups of X, it suffices to find a description of subgroups each of which is $\mathfrak{P}(\mathcal{C})'$-isolated but not $\mathcal{C}$-separable in X. Let us call such subgroups $\mathcal{C}$-*defective*.

In this paper, for a given root class of groups $\mathcal{C}$ and for the (ordinary) free product G of a family of groups, we study the $\mathcal{C}$-separability of subgroups of G. A free product almost always contains a non-abelian free subgroup and becomes a free group when its factors are all infinite cyclic. Therefore, any criterion for the $\mathcal{C}$-separability of subgroups of G is necessarily a generalization of the corresponding result on free groups.

It is known that, if $\mathcal{C}$ is a root class of groups, then any $\mathfrak{P}(\mathcal{C})'$-isolated cyclic subgroup of a free group is $\mathcal{C}$-separable in this group [12, Proposition 5.3]. If we replace here the word "cyclic" by "finitely generated", then the resulting assertion does not hold: the corresponding counterexample is given in [13]. Moreover, at present, a criterion for the $\mathcal{C}$-separability of an arbitrary finitely generated subgroup of a free group is known only when $\mathcal{C}$ is the class of all finite groups [14]. Therefore, when studying free products of groups, it makes sense to impose additional conditions on the considered subgroups, which in the case of a free group are equivalent to the property of being cyclic. In this article, such a condition is "to satisfy a non-trivial identity".

The following theorem is the main result of the paper. For a given root class of groups $\mathcal{C}$ and for the free product G of a family of residually $\mathcal{C}$-groups, it completely solves the question of the $\mathcal{C}$-separability of a subgroup of G that satisfies a non-trivial identity. It makes sense to note here that, by Proposition 2.5 below, the free product G is residually a $\mathcal{C}$-group if and only if all its factors have this property.

Theorem. *Suppose that $\mathcal{C}$ is a root class of groups, G is the free product of residually $\mathcal{C}$-groups A_i ($i \in \mathcal{I}$), and H is a subgroup of G satisfying a non-trivial identity. Then, H is $\mathcal{C}$-defective in G if and only if it is conjugate to a $\mathcal{C}$-defective subgroup of the group A_i for some $i \in \mathcal{I}$.*

Thus, to get a description of $\mathcal{C}$-separable subgroups of G which satisfy a non-trivial identity, it is sufficient to have such a description for every free factor A_i ($i \in \mathcal{I}$). The next two corollaries follow directly from the above theorem.

Corollary 1. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of residually $\mathcal{C}$-groups A_i ($i \in \mathcal{I}$). Suppose also that $\{\mathcal{V}_j | j \in \mathcal{J}\}$ is a family of varieties of groups, each $\mathcal{V}_j$ ($j \in \mathcal{J}$) is distinct from the variety of all groups, and $\mathcal{V} = \bigcup_{j \in \mathcal{J}} \mathcal{V}_j$. Then, a $\mathcal{V}$-subgroup H of the group G is $\mathcal{C}$-defective in this group if and only if it is conjugate to a $\mathcal{C}$-defective $\mathcal{V}$-subgroup of the group A_i for some $i \in \mathcal{I}$. In particular, if every group A_i ($i \in \mathcal{I}$) has the property of $\mathcal{C}$-separability of all $\mathfrak{P}(\mathcal{C})'$-isolated $\mathcal{V}$-subgroups, then the free product G also has this property.*

Corollary 2. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of residually $\mathcal{C}$-groups A_i ($i \in \mathcal{I}$). Then, a cyclic subgroup H of the group G is $\mathcal{C}$-defective in this group if and only if it is conjugate to a $\mathcal{C}$-defective cyclic subgroup of the group A_i for some $i \in \mathcal{I}$. In particular, if every group A_i ($i \in \mathcal{I}$) has the property of $\mathcal{C}$-separability of all $\mathfrak{P}(\mathcal{C})'$-isolated cyclic subgroups, then the free product G also has this property.*

Let us note that Corollary 2 generalizes Theorem 5 from [15], which asserts that the free product of π_c-groups is again a π_c-group (recall that a π_c-*group* is a group whose cyclic subgroups are all $\mathcal{F}$-separable, where $\mathcal{F}$ is the class of all finite groups). The analogue of this result for finitely generated subgroups was get in [16], and Corollary 1 complements it in some sense. The proof of the formulated theorem is given in Section 3. Section 2 contains a number of assertions necessary for this proof.

2. SOME AUXILIARY CONCEPTS AND RESULTS

Given a class of groups $\mathcal{C}$ and a group X, let us denote by $\mathcal{C}^*(X)$ the family of normal subgroups of X defined as follows: $N \in \mathcal{C}^*(X)$ if and only if $X/N \in \mathcal{C}$. It is obvious that the elements of $\mathcal{C}^*(X)$ are precisely the kernels of all possible homomorphisms of X onto groups from $\mathcal{C}$. Therefore, a subgroup Y of the group X is $\mathcal{C}$-separable in this group if and only if, for each element $x \in X \setminus Y$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $x \notin YN$.

Proposition 2.1 [17, Proposition 2]. *If $\mathcal{C}$ is a class of groups closed under taking subgroups and direct products of a finite number of factors, X is a group, and $Y, Z \in \mathcal{C}^*(X)$, then $Y \cap Z \in \mathcal{C}^*(X)$.*

Proposition 2.2. *Suppose that $\mathcal{C}$ is an arbitrary class of groups, X is a group, and Y is a subgroup of X. Then, the following statements hold.*

1. If the subgroup Y is $\mathcal{C}$-separable in X, then it is $\mathfrak{P}(\mathcal{C})'$-isolated in this group.

2. If the subgroup Y is $\mathfrak{P}(\mathcal{C})'$-isolated in X and Z is a $\mathfrak{P}(\mathcal{C})'$-isolated subgroup of Y, then Z is $\mathfrak{P}(\mathcal{C})'$-isolated in X.

Proof. The first statement coincides with Proposition 4.1 from [12], while the second one follows easily from the definition of a $\mathfrak{P}(\mathcal{C})'$-isolated subgroup.   □

Recall that a subgroup Y of a group X is said to be a *retract* of this group if there exists a homomorphism $\sigma \colon X \to Y$ acting identically on Y. This homomorphism is usually referred to as a *retracting* one.

Proposition 2.3 [18, Proposition 3]. *Let $\mathcal{C}$ be a class of groups closed under taking subgroups and extensions. If X is a residually $\mathcal{C}$-group, then any retract of X is a $\mathcal{C}$-separable subgroup.*

In what follows, let $\mathrm{sgp}\{S\}$ denote the subgroup generated by a set S.

Proposition 2.4. *If G is the free product of groups A_i ($i \in \mathcal{I}$), then, for each non-empty subset $\mathcal{J} \subseteq \mathcal{I}$, the subgroup $A_{\mathcal{J}} = \mathrm{sgp}\{A_j | j \in \mathcal{J}\}$ is a free factor of G, serves as a retract of this group, and splits into the free product of the groups A_j ($j \in \mathcal{J}$).*

Proof. Consider the mapping of the generators of G to $A_{\mathcal{J}}$ that acts identically on the generators of the groups A_j ($j \in \mathcal{J}$) and takes the generators of the groups A_i ($i \in \mathcal{I} \setminus \mathcal{J}$) to the trivial element. Obviously, this mapping can be extended to a homomorphism, which is identical on the subgroup $A_{\mathcal{J}}$. Therefore, the latter is a retract of G. The other two statements can be easily proved by grouping the generators and relations in the presentation of the group G.   □

Let $\mathcal{C}$ be a root class of groups. It is easy to see that any subgroup of a residually $\mathcal{C}$-group is also residually a $\mathcal{C}$-group. Therefore, the next proposition completely solves the question of whether the free product of a family of groups is residually a $\mathcal{C}$-group.

Proposition 2.5 [19, Theorem 2]. *If $\mathcal{C}$ is a root class of groups, then the free product of any family of residually $\mathcal{C}$-groups is itself residually a $\mathcal{C}$-group.*

Let G be the free product of groups A and B. Recall (see, for example, [20, Theorem 4.1]) that each element $g \in G$ can be uniquely represented as a product $g = g_1 g_2 \ldots g_n$, where $n \geqslant 0$, $g_1, g_2, \ldots, g_n \in (A \cup B) \setminus \{1\}$, and, for $n > 1$, no two adjacent factors g_i and g_{i+1} lie simultaneously in A or in B (it is assumed that, if $n = 0$, then the product $g_1 g_2 \ldots g_n$ is equal to 1). This product is referred to as the *reduced form* of the element g, and the number n is said to be the *length* of this element. The latter is denoted below by $\ell(g)$.

Obviously, every free product of two groups can be considered as the generalized free product of these groups with the trivial amalgamated subgroup (to avoid ambiguity, we always refer to a free product with an amalgamated subgroup as the *generalized* one). Therefore, the next proposition is a special case of Theorem 2.1 from [12].

Proposition 2.6. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of groups A and B. Suppose also that G has a homomorphism which maps it onto a $\mathcal{C}$-group and acts injectively on A and B. If a $\mathfrak{P}(\mathcal{C})'$-isolated subgroup of G satisfies a non-trivial identity, then it is $\mathcal{C}$-separable in G. In particular, G is residually a $\mathcal{C}$-group.*

Proposition 2.7 [21, Theorem 7]. *Let G be the generalized free product of groups A and B with an amalgamated subgroup U. If a subgroup H of G satisfies a non-trivial identity, then it is one of the following*:

a) subgroup of a conjugate of A or B;

b) the union of a countable non-descending sequence of subgroups, each of which has the form $gUg^{-1} \cap H$ for some $g \in G$;

c) an extension of the above union by an infinite cyclic group;

d) the generalized free product of groups $\tilde{A}$ and $\tilde{B}$ with an amalgamated subgroup $\tilde{U}$, where $\tilde{A}, \tilde{B} \in \{g_a A g_a^{-1} \cap H | g_a \in G\} \cup \{g_b B g_b^{-1} \cap H | g_b \in G\}$, $\tilde{U} = gUg^{-1} \cap H$ for some $g \in G$, and $[\tilde{A} : \tilde{U}] = 2 = [\tilde{B} : \tilde{U}]$.

Proposition 2.8. *Let G be the free product of groups A and B. If a subgroup H of G satisfies a non-trivial identity, then one and only one of the following statements holds.*

1. The subgroup H is conjugate to a subgroup of A or of B.

2. The subgroup H is infinite cyclic and is conjugate to no subgroup lying in $A \cup B$.

3. The subgroup H is the non-abelian split extension of an infinite cyclic group by a group of order 2 and is conjugate to no subgroup lying in $A \cup B$.

Proof. Indeed, let us consider the group G as the generalized free product of the groups A and B with the trivial amalgamated subgroup U. Then, by Proposition 2.7, the subgroup H is one of the following:

(a) a subgroup of a conjugate of A or B;

(b) a trivial group;

(c) an infinite cyclic group;

(d) the free product of groups $\tilde{A}$ and $\tilde{B}$ of order 2.

Suppose that the last possibility is realized and the symbols a and b denote the generators of the groups $\tilde{A}$ and $\tilde{B}$, respectively. Since, for each $k \geqslant 1$, the element $(ab)^k$ has a reduced form of length $2k$ in the free product H, the cyclic subgroup Z generated by ab is infinite. It follows from the equalities $a^{-1} = a$, $b^{-1} = b$, $(ab)^{-1} = ba$, and $a^{-1}(ab)a = ba = b^{-1}(ab)b$ that the subgroup Z is normal in H, but does not lie in the center of this group. It is also obvious that $H = \tilde{A}Z$ and $\tilde{A} \cap Z = 1$. Hence, H is the non-abelian split extension of the infinite cyclic group Z by the group $\tilde{A}$ of order 2. $\quad\square$

Given a class of groups $\mathcal{C}$, let us say that a group X is $\mathcal{C}$-*quasi-regular* with respect to its subgroup Y if, for each subgroup $M \in \mathcal{C}^*(Y)$, there exists a subgroup $N \in \mathcal{C}^*(X)$ such that $N \cap Y \leqslant M$. The next proposition is a special case of Corollary 2.4 from [12].

Proposition 2.9. *Suppose that $\mathcal{C}$ is a root class of groups, G is the generalized free product of residually $\mathcal{C}$-groups A and B with an amalgamated subgroup U, and H is a cyclic subgroup of G which is conjugate to no subgroup lying in $A \cup B$. Suppose also that U is $\mathcal{C}$-separable in the groups A and B, while G is $\mathcal{C}$-quasi-regular with respect to the subgroups A and B. If the subgroup H is $\mathfrak{P}(\mathcal{C})'$-isolated in G, then it is $\mathcal{C}$-separable in this group.*

Proposition 2.10. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of residually $\mathcal{C}$-groups A and B. Suppose also that H is a cyclic subgroup of G which is conjugate to no subgroup lying in $A \cup B$. If H is $\mathfrak{P}(\mathcal{C})'$-isolated in G, then it is $\mathcal{C}$-separable in this group.*

Proof. Suppose that $M \in \mathcal{C}^*(A)$ and $\varepsilon\colon A \to A/M$ is the natural homomorphism. Since A is a retract of G by Proposition 2.4, there exists a homomorphism $\sigma\colon G \to A$ which acts identically on A. If $N = \ker \sigma\varepsilon$, then $N \in \mathcal{C}^*(G)$ and $N \cap A = M$, as it is easy to see. Therefore, the group G is $\mathcal{C}$-quasi-regular with respect to A. Its $\mathcal{C}$-quasi-regularity with respect to B can be proved similarly. Since A and B are residually $\mathcal{C}$-groups, their trivial subgroups are $\mathcal{C}$-separable. Thus, the desired assertion follows from Proposition 2.9 if G is considered as the generalized free product of the groups A and B with the trivial amalgamated subgroup. $\quad\square$

3. PROOF OF THEOREM

Proposition 3.1. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of residually $\mathcal{C}$-groups A and B. Suppose also that H is a subgroup of G, this subgroup is the non-abelian split extension of an infinite cyclic group by a group of order 2 and is conjugate to no subgroup lying in $A \cup B$. If H is $\mathfrak{P}(\mathcal{C})'$-isolated in G, then it is $\mathcal{C}$-separable in this group.*

Proof. As it is known (see, for example, [20, Corollary 4.1.4]), every element of finite order of the group G is conjugate to an element of A or of B. Therefore, we may replace the subgroup H by its conjugate, rename the free factors A and B, if necessary, and assume further that H is the non-abelian split extension of an infinite cyclic subgroup $Z \leqslant G$ by a subgroup $Y \leqslant A$ of order 2. Let $Y = \{1, y\}$, and let z be a generator of Z. Then, $y^{-1}zy = z^{-1}$. We claim that the subgroup Z is conjugate to no subgroup contained in $A \cup B$.

Let, on the contrary, $x^{-1}Zx \subseteq A \cup B$ for some $x \in G$, and let $x_1 x_2 \ldots x_n$ be the reduced form of x. We will argue by induction on n. If $n = 0$ and $Z \leqslant A$, then $H = YZ \leqslant A$ despite the condition

of the proposition. If $n = 0$ and $Z \leqslant B$, then the product $y^{-1}zyz$ is the reduced form of the element $y^{-1}zyz$, and we get a contradiction with the equality $y^{-1}zy = z^{-1}$.

Suppose that $n \geqslant 1$, $x^{-1}Zx \leqslant A$, and, hence, $z = x_1 x_2 \ldots x_n a x_n^{-1} \ldots x_2^{-1} x_1^{-1}$ for some $a \in A \setminus \{1\}$. If $x_n \in A$ and $x' = x_1 x_2 \ldots x_{n-1}$, then $(x')^{-1}Zx' \leqslant A$ and the inductive hypothesis can be used. Let $x_n \in B$. In this case, the product $x_1 x_2 \ldots x_n a^{\pm 1} x_n^{-1} \ldots x_2^{-1} x_1^{-1}$ is the reduced form of the element $z^{\pm 1}$, and this element is of length $2n + 1$. If $x_1 \in B$, we have $\ell(y^{-1}zy) = 2n + 3 > 2n + 1 = \ell(z^{-1})$, which is impossible. Since the reduced form of the element $y^{-1}zy = z^{-1}$ is unique, it follows from the inclusion $x_1 \in A$ that $y^{-1}x_1 = x_1$ and $y = 1$, which is also impossible. The same arguments can be used in the case where $n \geqslant 1$ and $x^{-1}Zx \leqslant B$.

Thus, Z is conjugate to no subgroup contained in $A \cup B$. Since A is residually a $\mathcal{C}$-group, the inclusion $Y \leqslant A$ implies that $2 \in \mathfrak{P}(\mathcal{C})$. Therefore, Z is $\mathfrak{P}(\mathcal{C})'$-isolated in the subgroup H. The latter is $\mathfrak{P}(\mathcal{C})'$-isolated in G. Hence, Z is also $\mathfrak{P}(\mathcal{C})'$-isolated in G by Proposition 2.2 and is $\mathcal{C}$-separable in this group by Proposition 2.10.

To prove the $\mathcal{C}$-separability of H, let us fix an element $g \in G \setminus H$ and find a subgroup $N \in \mathcal{C}^*(G)$ satisfying the relation $g \notin HN$. Since $g \notin H = Z \cup yZ$, we have $g \notin Z$ and $y^{-1}g \notin Z$. It follows from the $\mathcal{C}$-separability of Z and Proposition 2.1 that there exists a subgroup $N \in \mathcal{C}^*(G)$ such that $g \notin ZN$ and $y^{-1}g \notin ZN$. If $g \in HN = ZN \cup yZN$, then $g \in ZN$ or $y^{-1}g \in ZN$ despite the choice of N. Therefore, the subgroup N is the desired one. $\qquad \square$

Proposition 3.2. *Suppose that $\mathcal{C}$ is a root class of groups and G is the free product of residually $\mathcal{C}$-groups A and B. Suppose also that H is a subgroup of G which is contained in $A \cup B$ and satisfies a non-trivial identity. If an element $g \in G$ and the subgroup H do not lie in the same free factor, then there exists a homomorphism σ of G onto a group from $\mathcal{C}$ such that $g\sigma \notin H\sigma$.*

Proof. Without lost of generality, we may assume that $H \leqslant A$. Let $g_1 g_2 \ldots g_n$ be the reduced form of g. Then $n \geqslant 1$ and $g_1 \in B$ if $n = 1$. For each $i \in \{1, 2, \ldots, n\}$, let us define subgroups M_i and N_i as follows. If $g_i \in A$, we put $N_i = B$ and find a subgroup $M_i \in \mathcal{C}^*(A)$ satisfying the relation $g_i \notin M_i$ (such a subgroup certainly exists because A is residually a $\mathcal{C}$-group). Similarly, if $g_i \in B$, we take A as M_i and choose a subgroup $N_i \in \mathcal{C}^*(B)$ which does not contain g_i.

Let $M = \bigcap_{i=1}^n M_i$ and $N = \bigcap_{i=1}^n N_i$. Then, A/M, $B/N \in \mathcal{C}$ by Proposition 2.1 and, for any $i \in \{1, 2, \ldots, n\}$, $g_i M \neq 1$ if $g_i \in A$, and $g_i N \neq 1$ if $g_i \in B$. It is easy to see that the natural homomorphisms $A \to A/M$ and $B \to B/N$ can be extended to a homomorphism ρ of the group G onto the free product $G_{M,N}$ of the groups A/M and B/N. It is also obvious that the product $(g_1\rho)(g_2\rho)\ldots(g_n\rho)$ is the reduced form of the element $g\rho$ and $g_1\rho \in B/N$ if $n = 1$. Therefore, $g\rho \notin H\rho$.

Since $A\rho = A/M \in \mathcal{C}$ and $B\rho = B/N \in \mathcal{C}$, the subgroup $H\rho$ is $\mathcal{C}$-separable in the group $A\rho$ and, by Proposition 2.5, the free product $G_{M,N}$ is residually a $\mathcal{C}$-group. The subgroup $A\rho$ is a retract of $G_{M,N}$ by Proposition 2.4 and is $\mathcal{C}$-separable in this group by Proposition 2.3. Hence, the subgroup $H\rho$ is $\mathfrak{P}(\mathcal{C})'$-isolated in the group $A\rho$, the subgroup $A\rho$ is $\mathfrak{P}(\mathcal{C})'$-isolated in the group $G_{M,N}$, and the subgroup $H\rho$ is $\mathfrak{P}(\mathcal{C})'$-isolated in the group $G_{M,N}$ by Proposition 2.2. It is also clear that the identity mappings of the groups $A\rho$ and $B\rho$ induce a homomorphism of $G_{M,N}$ onto the direct product $A\rho \times B\rho$, which belongs to $\mathcal{C}$. Therefore, the subgroup $H\rho$ is $\mathcal{C}$-separable in the group $G_{M,N}$ by Proposition 2.6 and the homomorphism ρ can be extended to the desired one. $\qquad \square$

Proposition 3.3. *Suppose that $\mathcal{C}$ is a root class of groups, G is the free product of residually $\mathcal{C}$-groups A_1, A_2, \ldots, A_n $(n \geqslant 1)$, and H is a $\mathfrak{P}(\mathcal{C})'$-isolated subgroup of G satisfying a non-trivial identity. If, for any $i \in \{1, 2, \ldots, n\}$, the subgroup H is conjugate to no $\mathcal{C}$-defective subgroup of A_i, then it is $\mathcal{C}$-separable in G and, therefore, is not $\mathcal{C}$-defective in this group.*

Proof. We will argue by induction on n. The proposition is trivial if $n = 1$, and we assume further that $n > 1$. By Proposition 2.4, the group G is the free product of the groups A_n and $B = \mathrm{sgp}\{A_1, A_2, \ldots, A_{n-1}\}$, while the latter is the free product of the groups A_1, A_2, \ldots, A_{n-1}. Since A_1, A_2, \ldots, A_n are residually $\mathcal{C}$-groups, it follows from Proposition 2.5 that B and G are also residually $\mathcal{C}$-groups. This fact allows one to apply Propositions 2.8, 2.10, and 3.1 to the free product of the groups A_n and B. If Statement 2 or Statement 3 of Proposition 2.8 holds, then H is $\mathcal{C}$-separable in G. Therefore, we may replace this subgroup by its conjugate, if necessary, and assume further

that $H \leqslant A_n$ or $H \leqslant B$. To prove the $\mathcal{C}$-separability of H, we fix an element $g \in G \setminus H$ and find a homomorphism σ of the group G onto a group from $\mathcal{C}$ such that $g\sigma \notin H\sigma$. Consider three cases.

Case 1. $H \leqslant A_n$ and $g \in A_n$.

By Proposition 2.4, the free factor A_n is a retract of G. Obviously, the subgroup H is $\mathfrak{P}(\mathcal{C})'$-isolated in the group A_n and is conjugate in A_n to no $\mathcal{C}$-defective subgroup. Hence, it is not $\mathcal{C}$-defective in A_n and, therefore, is $\mathcal{C}$-separable in this group. It follows that the retracting homomorphism $G \to A_n$, which acts identically on the subgroup A_n, can be extended to the desired mapping.

Case 2. $H \leqslant B$ and $g \in B$.

Similarly, the subgroup H is $\mathfrak{P}(\mathcal{C})'$-isolated in B and, for any $i \in \{1, 2, \ldots, n-1\}$, is conjugate to no $\mathcal{C}$-defective subgroup of the group A_i. Hence, it is $\mathcal{C}$-separable in B by the inductive hypothesis, and the retracting homomorphism $G \to B$ can be extended to the desired one.

Case 3. Either $H \leqslant A_n$ and $g \notin A_n$, or $H \leqslant B$ and $g \notin B$.

In this case, the existence of the required homomorphism is guaranteed by Proposition 3.2. $\qquad\square$

Proof of Theorem. *Necessity.* Let us argue by contradiction. If the subgroup H is not $\mathfrak{P}(\mathcal{C})'$-isolated in G, then it is not $\mathcal{C}$-defective in this group by the definition of the latter property. Therefore, we may assume that H is $\mathfrak{P}(\mathcal{C})'$-isolated in G and, for each $i \in \mathcal{I}$, is conjugate to no $\mathcal{C}$-defective subgroup of A_i. Let us show that H is then $\mathcal{C}$-separable in G and, hence, is not $\mathcal{C}$-defective in this group. If the set $\mathcal{I}$ is finite, the $\mathcal{C}$-separability of H follows from Proposition 3.3. Therefore, $\mathcal{I}$ can be assumed to be infinite. As above, to prove the $\mathcal{C}$-separability of H, we fix an element $g \in G \setminus H$ and find a homomorphism σ of G onto a group from $\mathcal{C}$ which satisfies the condition $g\sigma \notin H\sigma$.

Let $\mathcal{J}$ be a finite subset of $\mathcal{I}$ such that $g \in \mathrm{sgp}\{A_j | j \in \mathcal{J}\}$. Let also the symbols A and B denote the subgroups $\mathrm{sgp}\{A_j | j \in \mathcal{J}\}$ and $\mathrm{sgp}\{A_i | i \in \mathcal{I} \setminus \mathcal{J}\}$, respectively. By Proposition 2.4, the group G splits into the free product of the groups A and B, which in turn are the free products of the subgroups generating them. It follows that A, B, and G are residually $\mathcal{C}$-groups by Proposition 2.5. Let us apply Propositions 2.8, 2.10, and 3.1 to the group G considered as the free product of the groups A and B. As in the proof of Proposition 3.3, if Statement 2 or Statement 3 of Proposition 2.8 holds, then H is $\mathcal{C}$-separable in G. If H is conjugate to a subgroup of B and $\rho \colon G \to A$ is the retracting homomorphism, then $g\rho = g \neq 1 = H\rho$ and, since A is residually a $\mathcal{C}$-group, ρ can be extended to the desired mapping. Therefore, we may assume further that $x^{-1}Hx \leqslant A$ for some $x \in G$ and $x_1 x_2 \ldots x_n$ is the reduced form of the element x in the free product of A and B.

Let us use induction on n. If $n = 0$, then $H \leqslant A$ and the retracting homomorphism $G \to A$ can be extended to the desired one due to Proposition 3.3. If $n \geqslant 1$, $x_n \in A$, and $x' = x_1 x_2 \ldots x_{n-1}$, then $(x')^{-1}Hx' \leqslant A$ and the required homomorphism exists by the inductive hypothesis. Suppose that $n \geqslant 1$, $x_n \in B$, $\tilde{H} = x^{-1}Hx$, and $\tilde{g} = x^{-1}gx$. If $x_1 \in B$, then $\ell(\tilde{g}) = 2n + 1 \geqslant 3$. If $x_1 \in A$, it follows from the relations $x_n \in B$ and $g \notin H$ that $n \geqslant 2$, $x_1^{-1}gx_1 \neq 1$, and hence $\ell(\tilde{g}) = 2n - 1 \geqslant 3$. Thus, in both cases, $\tilde{H} \leqslant A$, $\tilde{g} \notin A$, and the existence of the required homomorphism is ensured by Proposition 3.2.

Sufficiency. Suppose that $x^{-1}Hx \leqslant A_i$ for some $x \in G$ and $i \in \mathcal{I}$, and the subgroup $\tilde{H} = x^{-1}Hx$ is $\mathcal{C}$-defective in the group A_i. Then, $\tilde{H}$ is $\mathfrak{P}(\mathcal{C})'$-isolated in A_i and there exists an element $a \in A_i \setminus \tilde{H}$ such that $a\sigma_i \in \tilde{H}\sigma_i$ for every homomorphism σ_i of the group A_i onto a group from $\mathcal{C}$. Let σ be an arbitrary homomorphism of G onto a $\mathcal{C}$-group. Since the class $\mathcal{C}$ is closed under taking subgroups, the restriction of σ to the subgroup A_i maps the latter onto a group from $\mathcal{C}$. Therefore, $a\sigma \in \tilde{H}\sigma$, $xax^{-1}\sigma \in H\sigma$, and, since $xax^{-1} \notin H$, the subgroup H is not $\mathcal{C}$-separable in G.

As noted above, G is residually a $\mathcal{C}$-group. Hence, the subgroup A_i is $\mathcal{C}$-separable in G due to Propositions 2.4 and 2.3. This fact and Proposition 2.2 imply that the subgroups A_i, $\tilde{H}$, and H are $\mathfrak{P}(\mathcal{C})'$-isolated in the group G. Thus, the subgroup H is $\mathcal{C}$-defective in G, as required. $\qquad\square$

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CONFLICT OF INTEREST

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